INTERNATIONAL JOURNAL OF
ENERGY AND ENVIRONMENT

Volume 6, Issue 6, 2015 pp.537-552
Journal homepage: www.IJEE.IEEFoundation.org

Optimal fundamental characteristic of a quantum harmonic oscillator Carnot refrigerator with multi-irreversibilities

Xiaowei Liu[1,2,3], Lingen Chen[1,2,3*], Feng Wu[1,2,3,4], Fengrui Sun[1,2,3]

[1] Institute of Thermal Science and Power Engineering, Naval University of Engineering, Wuhan 430033.
[2] Military Key Laboratory for Naval Ship Power Engineering, Naval University of Engineering, Wuhan 430033.
[3] College of Power Engineering, Naval University of Engineering, Wuhan 430033.
[4] School of Science, Wuhan Institute of Technology, Wuhan 430074, P. R. China

Abstract
The optimal performance of an irreversible quantum Carnot refrigerator with working medium consisting of many non-interacting harmonic oscillators is investigated in this paper. The quantum refrigerator cycle is composed of two isothermal processes and two irreversible adiabatic processes, and the irreversibilities of heat resistance, internal friction and bypass heat leakage are considered. By using the quantum master equation, semi-group approach and finite time thermodynamics (FTT), this paper derives the cooling load and coefficient of performance (COP) of the quantum refrigeration cycle and provides detailed numerical examples. At high temperature limit, the cooling load versus COP characteristic curves are plotted, and effects of internal friction and bypass heat leakage on the optimal performance of the quantum refrigerator are discussed. Three special cases, i.e., endoreversible, frictionless and without bypass heat leakage, are discussed in brief.

Keywords: Finite time thermodynamics; Harmonic oscillator system; Quantum refrigeration cycle; Cooling load; COP.

1. Introduction
With rapid development in fields of aerospace, superconductivity application and infra-red techniques etc., demands of cryogenic technology increase greatly. By using the finite time thermodynamics [1-12] and considering quantum characteristic of the working medium, many researchers have investigated the performance of quantum cycles and obtained many meaningful results. Geva and Kosloff [13] introduced the dynamical semi-group approach of quantum mechanics and non-equilibrium statistical theory into the FTT, established an endoreversible quantum heat engine model with working medium consisting of many non-interacting spin-1/2 systems, and obtained the optimal performance of the quantum heat engine. Geva and Kosloff [14] established another endoreversible quantun Carnot heat engine model using many non-interacting harmonic oscillators as working meidum, and indicated that the optimal cycles of spin-1/2 and harmonic heat engines are not Carnot cycle. Then, Wu et al [15] established an endoreversible quantum harmonic Stirling heat engine model and investageted its optimal performance. Feldmann et al [16] investigated the optimal performance of an endoreversible quantum spin-1/2 Baryton heat engine. Wu et al [17] first established a quantum spin-1/2 Carnot refrigerator model and obtained the otpimal performance parameters and the optimal relation between the cooling load and COP of the

quantum refrigerator. Wu *et al* [18] established a quantum harnonic Carnot refrigerator model and obtained the optimal relation between the cooling load and COP of the quantum refrigerator. Besides quantum Carnot refrigeration cycles, Wu *et al* [19] and Lin *et al* [20] established endoreversible quantum harmonic Stirling [19] and Brayton [20] refrigerator models and obtained the optimal performance of these quantum refrigerators. He *et al* [21] investigated the optimal performance of an endoreversible quantum harmonic Brayotn refrigerator.

In the work mentioned above, the quantum cycles are endoreversible and the irreversibility of heat resistance is the sole irreversibility considered in the cycles. However, real heat devices are usually devices with internal and external irreversibilities. There are various sources of irreversibility, such as heat resistance, bypass heat leakage, dissipation processes inside the working medium, etc. Jin *et al* [22] introduced bypass heat leakage into exergoeconomic performance optimization of a quantum harmonic Carnot engine, and the bypass heat leakage arose from the thermal coupling between the hot and cold heat reservoirs. Feldmann and Kosloff [23] introduced internal friction in the performance investigation for a quantum spin-1/2 Brayton heat engine and refrigerator. Since then, some authors explored the origin of internal friction and investigated the effects of quantum friction on performance of quantum thermodynamic cycles [24-29]. Rezek and Kosloff [30] investigated the optimal performance of an irreversible harmonic Otto heat engine with internal friction and indicated that the irreversible loss in the quantum cycles was owed to finite rate of heat transfer and internal friction. The internal friction could be traced to the non-commutability of kinetic and potential energy of the working medium. He *et al* [31] established an irreversible quantum harmonic Otto refrigerator model and investigated the effects of internal friction on the optimal performance. By considering the irreversibilities of heat resistance and inherent regenerative loss, He *et al* [32] and Lin *et al* [33] investigated the optimal performance of irreversible spin-1/2 Ericsson refrigerator [32] and irreversible harmonic Striling refrigerator [33], respectively, and analyzed the effects of inherent regenerative loss on the optimal performance. Wu *et al* [34] established a general irreversible quantum harmonic Brayton refrigerator model, and obtained the optimal relationship between the dimensionless cooling load and the COP and the optimization region (or criteria). The effects of bypass heat leakage, irreversibility in two adiabatic processes and the quantum characteristic of the working fluid were discussed. Different from the internal friction introduced in Refs. [23, 29], an internal irreversible factor ϕ was used to describe the irreversibility inside the irreversible adiabatic processes in the quantum refrigeration cycle. Wu *et al* [35] established a general irreversible quantum spin-1/2 Ericsson refrigerator model with losses of heat resistance, bypass heat leakage and internal irreversibility, and derived the optimal relationship between the cooling load and COP for the irreversible quantum refrigerator. In particular, the performance characteristics of the cooler at the low temperature limit are discussed. By considering losses of heat resistance, internal friction and bypass heat leakage, Liu *et al* [36, 37] established models of general irreversible quantum Carnot heat engines with harmonic oscillators [36] and spin-1/2 systems [37], and investigated the optimal ecological performances of these quantum heat engines. The irreversibility in the adiabatic process was described by internal friction coefficient which was different from the internal irreversible factors used in Refs. [34, 35].

Based on Refs. [22, 23, 34, 35], the aim of this paper is to analyze and optimize the performance of an irreversible quantum Carnot refrigerator with irreversibilities of heat resistance, internal friction and bypass heat leakage. The working medium of the quantum refrigerator is consisting of many non-interacting harmonic oscillators. The quantum refrigeration cycle is composed of two isothermal processes and two irreversible adiabatic processes. By using the quantum master equation, semi-group approach and FTT, this paper will derive the cooling load and COP and provide detailed numerical examples. At high temperature limit, the cooling load versus COP characteristic curves will be plotted. The effects of internal friction and bypass heat leakage on the optimal performance will be discussed. Three special cases, that is, endoreversible case, frictionless case and the case without bypass heat leakage, will be are discussed in brief.

2. Quantum dynamics of a harmonic oscillator system

Consider a quantum system consisting of many non-interacting harmonic oscillators, according to quantum mechanics theory, the Hamiltonian $\hat{H}_S$ of this quantum system is given by [38, 39]

$$\hat{H}_S = \hbar\omega(t)\hat{N} = \hbar\omega(t)\hat{a}^{+}\hat{a} \tag{1}$$

where $\hat{a}^+$ and $\hat{a}$ are the Bosonic creation and annihilation operators, $\hbar$ is the reduced Planck's constant, $\hat{N} = \hat{a}^+\hat{a}$ is the number operator, and $\omega(t)$ is the frequency of the oscillator. Based on the quantum statistical theory, the population n of the oscillator at thermal equilibrium can be obtained from the Bose-Einstein distribution [38, 40]

$$n = 1\big/(e^{\hbar\omega/(k_\mathrm{B}T)} - 1) = 1\big/(e^{\hbar\omega\beta} - 1) \tag{2}$$

where $\beta = 1/(k_\mathrm{B}T)$, k_B is the Boltzmann constant and T is the absolute temperature. For convenience, the "temperature" will refer to β rather than T throughout this paper.
The internal energy of the harmonic oscillator system is given by

$$E_\mathrm{S} = \left\langle \hat{H}_\mathrm{S} \right\rangle = \hbar\omega(t)\left\langle \hat{N} \right\rangle = \hbar\omega(t)n \tag{3}$$

If there exists thermal coupling between the harmonic oscillator system and a heat reservoir (bath), the harmonic oscillator system becomes an open system. The total Hamiltonian of the system-bath is given by [41, 42]

$$\hat{H} = \hat{H}_\mathrm{S} + \hat{H}_\mathrm{SB} + \hat{H}_\mathrm{B} \tag{4}$$

where $\hat{H}_\mathrm{S}$, $\hat{H}_\mathrm{SB}$ and $\hat{H}_\mathrm{B}$ are Hamiltonians of the harmonic oscillator system, the system-bath interaction and the bath, respectively. The system-bath Hamiltonian is further assumed to be represented in the form of

$$\hat{H}_\mathrm{SB} = \sum_\alpha \hat{\Gamma}_\alpha \hat{Q}_\alpha \hat{B}_\alpha \tag{5}$$

where $\hat{Q}_\alpha$, $\hat{B}_\alpha$ and $\hat{\Gamma}_\alpha$ are operators of the harmonic oscillator system, the bath and the interaction strength. For an system operator $\hat{X}$, the effects of $\hat{H}_\mathrm{SB}$ and $\hat{H}_\mathrm{B}$ on the Hamiltonian are included in the Heisenberg equation as additional relaxation-type terms. In the Heisenberg picture, the motion of an operator is the master equation

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}_\mathrm{S}\, ,\, \hat{X}\right] + \frac{\partial\hat{X}}{\partial t} + L_\mathrm{D}(\hat{X}) \tag{6}$$

where $L_\mathrm{D}(\hat{X})$ is the dissipation term (the relaxation term) which arises from the thermal coupling between the harmonic oscillator system and heat reservoir.
Substituting $\hat{X} = \hat{H}_\mathrm{S} = \hbar\omega\hat{N}$ into the master equation (6) yields the rate of change of energy

$$\frac{\mathrm{d}E_\mathrm{S}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{H}_\mathrm{S} \right\rangle = \hbar\frac{\mathrm{d}\omega}{\mathrm{d}t}\left\langle \hat{N} \right\rangle + \hbar\omega\left\langle L_\mathrm{D}(\hat{N}) \right\rangle = \hbar n\dot{\omega} + \hbar\omega\dot{n} \tag{7}$$

In the right-hand side of equation (7), the first term

$$\hbar n\dot{\omega} = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{8}$$

Represents the energy level structure change and corresponds to instantaneous power, and the second term

$$\hbar\omega\dot{n} = \frac{\mathrm{d}Q}{\mathrm{d}t} = \dot{Q} \tag{9}$$

Represents the harmonic oscillator transitions between energy levels and corresponds to the instantaneous heat flow between the harmonic system and surrounding. The work and heat inexact differentials may be identified by

$$\mathrm{d}W = \hbar n \mathrm{d}\omega \tag{10}$$

$$\mathrm{d}Q = \hbar \omega \mathrm{d}n \tag{11}$$

For a harmonic oscillator system, equation (7) gives the time derivative of the first law of thermodynamics.

3. An irreversible harmonic oscillator Carnot refrigerator model

The irreversible harmonic oscillator Carnot refrigerator considered herein has the following constraints:

1. The working medium of the quantum refrigerator is modeled as a gas consisting of many non-interacting harmonic oscillators.
2. The quantum refrigerator operates between a hot reservoir B_{h} and a cold reservoir B_c. The two reservoirs are thermal phonon systems and at constant "temperatures" $\beta_{\mathrm{h}} = 1/(k_{\mathrm{B}}T_{\mathrm{h}})$ and $\beta_{\mathrm{c}} = 1/(k_{\mathrm{B}}T_{\mathrm{c}})$, respectively. The two heat reservoirs are infinitely large and their internal relaxations are very strong. Therefore, the two heat reservoirs are assumed to be in thermal equilibrium.
3. The $n-\omega$ diagram of an irreversible quantum Carnot refrigeration cycle is shown in Figure 1. The quantum refrigeration cycle is composed of two isothermal branches and two irreversible adiabatic branches.

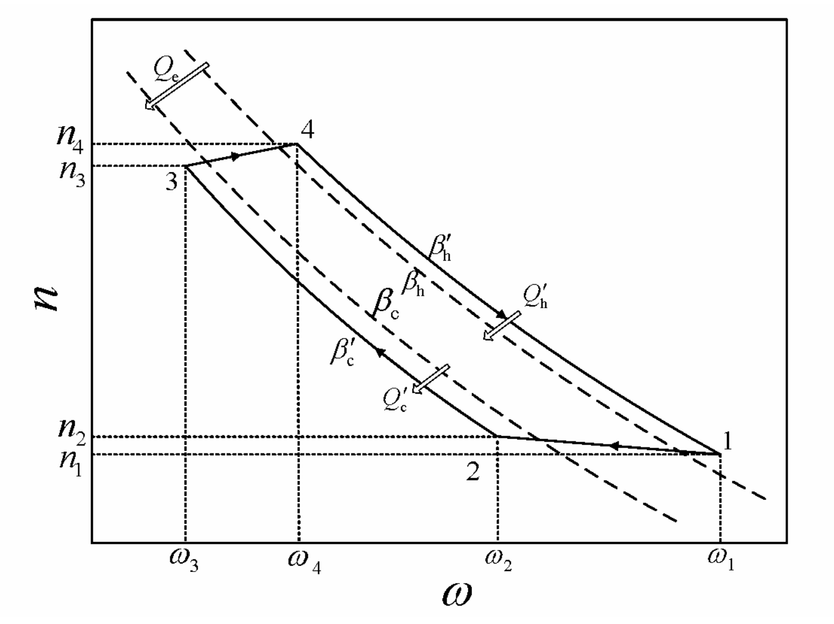


Figure 1. The $n-\omega$ diagram of an irreversible quantum harmonic Carnot refrigeration cycle

In the two isothermal processes, the working medium couples thermally to the heat reservoirs and exchanges heat with the heat reservoirs. The "temperatures" of the working medium in processes $4 \to 1$ and $2 \to 3$ are $\beta_{\mathrm{h}}' = 1/(k_{\mathrm{B}}T_{\mathrm{h}}')$ and $\beta_{\mathrm{c}}' = 1/(k_{\mathrm{B}}T_{\mathrm{c}}')$, respectively. The second law of thermodynamics requires $\beta_{\mathrm{c}} > \beta_{\mathrm{c}}' > \beta_{\mathrm{h}}' > \beta_{\mathrm{h}}$.

In the two adiabatic processes $1 \to 2$ and $3 \to 4$, there is no thermal coupling between the working medium and the hot reservoirs so that there is no heat exchange. Assume that the required time of the processes $3 \to 4$ and $1 \to 2$ are τ_{a} and τ_{b}, respectively, and the frequency of the oscillator changes linearly with time

$$\omega(t) = \omega(0) + \dot{\omega} t \tag{12}$$

According to quantum adiabatic theorem [39], rapid change of frequency causes quantum non-adiabatic phenomenon, and the harmonic population n becomes variable in the adiabatic process. The effect of non-adiabatic phenomenon on the performance of the quantum refrigerator is similar to that of internally dissipative friction in the classical analysis. It is therefore assumed that the non-adiabatic phenomenon can be described by an internal friction. One assumes that the internal friction forces a constant speed population change rate [23, 43]

$$\dot{n} = (\frac{\mu}{t'})^2 \tag{13}$$

where μ is friction coefficient and t' is the time spent on the corresponding adiabatic process. The population of harmonic oscillators in the adiabatic process may be expressed as

$$n(t) = n(0) + (\frac{\mu}{t'})^2 t \tag{14}$$

substituting $t = \tau_a$ and $t = \tau_b$ into equation (14) yields

$$n_2 = n_1 + \frac{\mu^2}{\tau_b}, \quad n_4 = n_3 + \frac{\mu^2}{\tau_a} \tag{15}$$

where $n_1 = 1/(e^{\hbar\beta_h'\omega_1} - 1)$, $n_2 = 1/(e^{\hbar\beta_c'\omega_2} - 1)$, $n_3 = 1/(e^{\hbar\beta_c'\omega_3} - 1)$ and $n_4 = 1/(e^{\hbar\beta_h'\omega_4} - 1)$ are the populations of harmonic oscillators at thermal equilibrium states 1, 2, 3 and 4, respectively. Using equation (15) yields

$$\omega_2 = \frac{1}{\hbar\beta_c'} \ln \frac{\tau_b e^{\hbar\beta_h'\omega_1} + \mu^2 (e^{\hbar\beta_h'\omega_1} - 1)}{\tau_b + \mu^2 (e^{\hbar\beta_h'\omega_1} - 1)} \tag{16}$$

$$\omega_4 = \frac{1}{\hbar\beta_h'} \ln \frac{\tau_a e^{\hbar\beta_c'\omega_3} + \mu^2 (e^{\hbar\beta_c'\omega_3} - 1)}{\tau_a + \mu^2 (e^{\hbar\beta_c'\omega_3} - 1)} \tag{17}$$

The works done on the system along processes $1 \rightarrow 2$ and $3 \rightarrow 4$ can be calculated from equations (7), (12) and (14), respectively

$$W_{12} = \hbar \int_0^{\tau_b} n d\omega = \hbar(\omega_2 - \omega_1)(n_1 + \frac{\mu^2}{2\tau_b}) + \hbar \frac{\mu^2 (\omega_1 + \omega_2)}{2\tau_b} \tag{18}$$

$$W_{34} = \hbar \int_0^{\tau_a} n d\omega = \hbar(\omega_4 - \omega_3)(n_3 + \frac{\mu^2}{2\tau_a}) + \hbar \frac{\mu^2 (\omega_4 + \omega_3)}{2\tau_a} \tag{19}$$

From equation (9), one can get that the second part of the right sides of equations (18) and (19) $\hbar \frac{\mu^2 (\omega_1 + \omega_2)}{2\tau_b}$ and $\hbar \frac{\mu^2 (\omega_4 + \omega_3)}{2\tau_a}$ are the heats generated on processes $1 \rightarrow 2$ and $3 \rightarrow 4$, respectively, and these parts of work are against the friction.

4. Besides heat resistance and internal friction, there exists bypass heat leakage between the hot and cold reservoirs. The bypass heat leakage arises from the thermal coupling action between the hot reservoir and cold reservoir by the working medium of the quantum refrigerator
5. The effect of Bose-Einstein condensation of the working medium (non-interacting harmonic oscillator system) is not considered in this paper, viz. $\beta_c' < \beta_e$, where $\beta_e = 1/(k_B T_e)$ and T_e is the critical temperature of Bose-Einstein condensation. The effect of relativity theory is not considered, too.

The model established in this paper is similar to the models of generalized irreversible Carnot refrigerator with classical working medium with several irreversibilities, such as heat resistance, internal irreversibility and bypass heat leakage [44-49].

4. Cycle period

According to quantum semi-group approach, the dissipation term in equation (7) becomes [14, 41, 42]

$$\hat{L}_{\mathrm{D}}(\hat{X}) = \sum_{\alpha} \gamma_{\alpha} (\hat{Q}_{\alpha}^{+} \left[\hat{X}, \hat{Q}_{\alpha} \right] + \left[\hat{Q}_{\alpha}^{+}, \hat{X} \right] \hat{Q}_{\alpha}) \tag{20}$$

where $\hat{Q}_{\alpha}$ and $\hat{Q}_{\alpha}^{+}$ are operators in the Hilbert space of the harmonic oscillator system and Hermitian conjugate, and γ_{α} is phenomenological positive coefficient.

Substituting $\hat{Q}_{\alpha} = \hat{a}$, $\hat{Q}_{\alpha}^{+} = \hat{a}^{+}$ into equation (7) yields

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} = i\omega \left[\hat{a}^{+}\hat{a}, \hat{X} \right] + \frac{\partial \hat{X}}{\partial t} + \gamma_{+} (\hat{a} \left[\hat{X}, \hat{a}^{+} \right] + \left[\hat{a}, \hat{X} \right] \hat{a}^{+}) + \gamma_{-} (\hat{a}^{+} \left[\hat{X}, \hat{a} \right] + \left[\hat{a}^{+}, \hat{X} \right] \hat{a}) \tag{21}$$

Substituting $\hat{X} = \hat{N} = \hat{a}^{+}\hat{a}$ into equation (21) and using $\left[\hat{a}, \hat{a}^{+} \right] = 1$, $\left[\hat{N}, \hat{a}^{+} \right] = \hat{a}^{+}$ and $\left[\hat{N}, \hat{a} \right] = -\hat{a}$ yields the time evolution of harmonic oscillator population

$$\dot{n} = \frac{\mathrm{d}\left\langle \hat{N} \right\rangle}{\mathrm{d}t} = -2(\gamma_{-} - \gamma_{+})n + 2\gamma_{+} \tag{22}$$

Solving equation (22) yields

$$n = n_{\mathrm{e}} + (n_0 - n_{\mathrm{e}})\,\mathrm{e}^{-2(\gamma_{-} - \gamma_{+})t} \tag{23}$$

where n_0 is the initial value of n and $n_{\mathrm{e}} = \gamma_{+} / (\gamma_{-} - \gamma_{+})$ is the asymptotic value of n. This asymptotic population of oscillators must correspond to the value at thermal equilibrium state $n_{\mathrm{e}} = 1/(\mathrm{e}^{\hbar\beta_j\omega} - 1)$, where $j = \mathrm{h}, \mathrm{c}$ correspond to isothermal processes $4 \to 1$ and $2 \to 3$, respectively. Comparison of the two expressions of n_{e} yields

$$\gamma_{+} = \mathrm{a}\mathrm{e}^{\mathrm{q}\hbar\beta_j\omega}, \quad \gamma_{-} = \mathrm{a}\mathrm{e}^{(1+\mathrm{q})\hbar\beta_j\omega} \tag{24}$$

where both a and q are two constants. $\gamma_{+}, \gamma_{-} > 0$ requires $\mathrm{a} > 0$. If $\beta_j\omega \to \infty$, $\gamma_{+} \to 0$ and $\gamma_{-} \to \infty$ hold, it requires $0 > \mathrm{q} > -1$.

Substituting equations (24) into equation (22) yields

$$\dot{n} = \frac{\mathrm{d}\left\langle \hat{\mathrm{N}} \right\rangle}{\mathrm{d}t} = -2\mathrm{a}\mathrm{e}^{\mathrm{q}\hbar\beta_j\omega}[(\mathrm{e}^{\hbar\beta_j\omega} - 1)n - 1] \tag{25}$$

The times of isothermal processes $4 \to 1$ and $2 \to 3$ are, respectively

$$\tau_{\mathrm{h}} = \int_{\omega_4}^{\omega_1} \frac{\mathrm{d}n/\mathrm{d}\omega}{\dot{n}} \mathrm{d}\omega = \frac{1}{2\mathrm{a}} \int_{\ln[(n_4+1)/n_4]}^{\ln[(n_1+1)/n_1]} \frac{\mathrm{d}m_{\mathrm{h}}}{\mathrm{e}^{\mathrm{q}\alpha_{\mathrm{h}}m_{\mathrm{h}}} (\mathrm{e}^{\alpha_{\mathrm{h}}m_{\mathrm{h}}} - \mathrm{e}^{m_{\mathrm{h}}})(1 - \mathrm{e}^{-m_{\mathrm{h}}})} \tag{26}$$

$$\tau_{\mathrm{c}} = \int_{\omega_2}^{\omega_3} \frac{\mathrm{d}n/\mathrm{d}\omega}{\dot{n}} \mathrm{d}\omega = \frac{1}{2\mathrm{a}} \int_{\ln[(n_3+1)/n_3]}^{\ln[(n_2+1)/n_2]} \frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{e}^{\mathrm{q}\alpha_{\mathrm{c}}m_{\mathrm{c}}} (\mathrm{e}^{m_{\mathrm{c}}} - \mathrm{e}^{\alpha_{\mathrm{c}}m_{\mathrm{c}}})(1 - \mathrm{e}^{-m_{\mathrm{c}}})} \tag{27}$$

where $m_{\mathrm{h}} = \hbar\beta_{\mathrm{h}}'\omega$, $m_{\mathrm{c}} = \hbar\beta_{\mathrm{c}}'\omega$, $\alpha_{\mathrm{h}} = \beta_{\mathrm{h}} / \beta_{\mathrm{h}}' > 1$ and $\alpha_{\mathrm{c}} = \beta_{\mathrm{c}} / \beta_{\mathrm{c}}' < 1$.

The cycle period is given by

$$\begin{aligned} \tau &= \tau_{\mathrm{a}} + \tau_{\mathrm{b}} + \tau_{\mathrm{h}} + \tau_{\mathrm{c}} \\ &= \frac{1}{2\mathrm{a}} \int_{\ln[(n_4+1)/n_4]}^{\ln[(n_1+1)/n_1]} \frac{\mathrm{d}m_{\mathrm{h}}}{\mathrm{e}^{\mathrm{q}\alpha_{\mathrm{h}}m_{\mathrm{h}}} (\mathrm{e}^{\alpha_{\mathrm{h}}m_{\mathrm{h}}} - \mathrm{e}^{m_{\mathrm{h}}})(1 - \mathrm{e}^{-m_{\mathrm{h}}})} + \frac{1}{2\mathrm{a}} \int_{\ln[(n_3+1)/n_3]}^{\ln[(n_2+1)/n_2]} \frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{e}^{\mathrm{q}\alpha_{\mathrm{c}}m_{\mathrm{c}}} (\mathrm{e}^{m_{\mathrm{c}}} - \mathrm{e}^{\alpha_{\mathrm{c}}m_{\mathrm{c}}})(1 - \mathrm{e}^{-m_{\mathrm{c}}})} + \tau_{\mathrm{a}} + \tau_{\mathrm{b}} \end{aligned} \tag{28}$$

5. Cooling load and COP

Using equation (9), one can get the amounts of heat exchange between the working medium and the hot reservoir in isothermal processes $4 \to 1$ and $2 \to 3$, respectively

$$Q'_{\rm h} = -\hbar\int_{n_4}^{n_1} \omega dn = \frac{1}{\beta'_{\rm h}}(n_1 \ln\frac{n_1}{1+n_1} - n_4 \ln\frac{n_4}{1+n_4} + \ln\frac{1+n_4}{1+n_1}) \tag{29}$$

$$Q'_{\rm c} = \hbar\int_{n_2}^{n_3} \omega dn = \frac{1}{\beta'_{\rm c}}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2}) \tag{30}$$

The working medium system releases heat in the process $4 \to 1$ so that there is a minus before the integral. From equation (8), one can get the works done on the system along these processes, respectively

$$W_{41} = \hbar\int_{\omega_4}^{\omega_1} n d\omega = \frac{1}{\beta'_{\rm h}}\ln\frac{n_4}{n_1} + \hbar(\omega_4 - \omega_1) \tag{31}$$

$$W_{23} = \hbar\int_{\omega_2}^{\omega_3} n d\omega = \frac{1}{\beta'_{\rm c}}\ln\frac{n_2}{n_3} + \hbar(\omega_2 - \omega_3) \tag{32}$$

Similar to the calculation of the heat flow between the working medium and heat reservoirs, one can calculate the bypass heat leakage. Similar to $\dot{n}$, derivative of the population of cold reservoir $n_{\rm c}$ can be derived as follows at the condition of small thermal disturbance

$$\dot{n}_{\rm c} = -2{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}} - 1)n_{\rm c} - 1] \tag{33}$$

where $\omega_{\rm c}$ is the frequency of the thermal phonons of the cold reservoir, c and λ are two constants.
Using equations (9) and (33) yields the heat flow from hot reservoir to coal reservoir (i.e. rate of bypass heat leakage) [22]

$$\dot{Q}_{\rm e} = C_{\rm e}\hbar\omega_{\rm c}\dot{n}_{\rm c} = 2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1 - ({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}} - 1)n_{\rm c}] \tag{34}$$

where $C_{\rm e}$ is a dimensionless factor which describes the magnitude of the bypass heat leakage. According to the refrigerator model, the hot and cold reservoirs are assumed to be in thermal equilibrium and $\omega_{\rm c}$ may be assumed to be a constant. The bypass heat leakage quantity per cycle is given by

$$Q_{\rm e} = \dot{Q}_{\rm e}\tau = 2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1 - ({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}} - 1)n_{\rm c}]\tau \tag{35}$$

Combining equations (28) and (30) with equation (35) yields the cooling load

$$\begin{aligned} R &= Q_{\rm c}/\tau \\ &= \frac{1}{\beta'_{\rm c}}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2})\tau^{-1} - 2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1 - ({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}} - 1)n_{\rm c}] \end{aligned} \tag{36}$$

where $Q_{\rm c} = Q'_{\rm c} - Q_{\rm e}$ is the total heat released by the cold reservoir. Combining equations (29) and (30) with equation (35) yields the COP

$$\begin{aligned} \varepsilon &= Q_{\rm c}/Q_{\rm h} \\ &= \frac{\frac{1}{\beta'_{\rm c}}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2}) - 2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1 - ({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}} - 1)n_{\rm c}]\tau}{\frac{1}{\beta'_{\rm h}}(n_1 \ln\frac{n_1}{1+n_1} - n_4 \ln\frac{n_4}{1+n_4} + \ln\frac{1+n_4}{1+n_1}) - \frac{1}{\beta'_{\rm c}}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2})} \end{aligned} \tag{37}$$

where $Q_h = Q'_h - Q_e$ is the total heat absorbed by hot reservoir.

From equations (36) and (37), one can see that both the cooling load R and COP ε are functions of β'_h and β'_c for given β_h, β_c, β_0, q, a, c, λ, ω_1, ω_3, ω_h, C_e and μ. The integral in the denominators of equations (36) and (37) is unable to evaluate in closed form for the general case, therefore, one can not obtain the fundamental relation between the cooling load and COP analytically. Using numerical calculations, one can plot three-dimensional diagrams of dimensionless cooling load ($R/R_{\max,\mu=0,C_e=0}$, β'_h, β'_c) and COP (ε, β'_h, β'_c) as shown in Figures 2 and 3, where $R_{\max,\mu=0,C_e=0}$ is the maximum cooling load for endoreversible case. The parameters used in the numerical calculations are $a = c = 2$, $q = \lambda = -0.5$, $\beta_h = 1/(2k_B)$, $\beta_c = 1/k_B$, $\tau_a = \tau_b = 0.01$, $\omega_1 = 5\times10^{10}$, $\omega_3 = 8\times10^9$, $\omega_c = 1\times10^{10}$, $\mu = 0.01$, and $C_e = 0.01$. Figure 2 shows that there exist optimal "temperatures" β'_h and β'_c which lead to a maximum dimensionless cooling load for given "temperatures" of hot and cold reservoirs and other parameters. Affected by the internal friction and bypass heat leakage, the maximum dimensionless cooling load $(R/R_{\max,\mu=0,C_e=0})_{\max} < 1$. Figure 3 shows that there also exist optimal "temperatures" β'_h and β'_c which lead to a maximum COP with nonzero corresponding dimensionless cooling load when there exists a bypass heat leakage, and the optimal "temperature" β'_h (or β'_c) is close to the heat reservoir "temperature" β_h (or β_c).

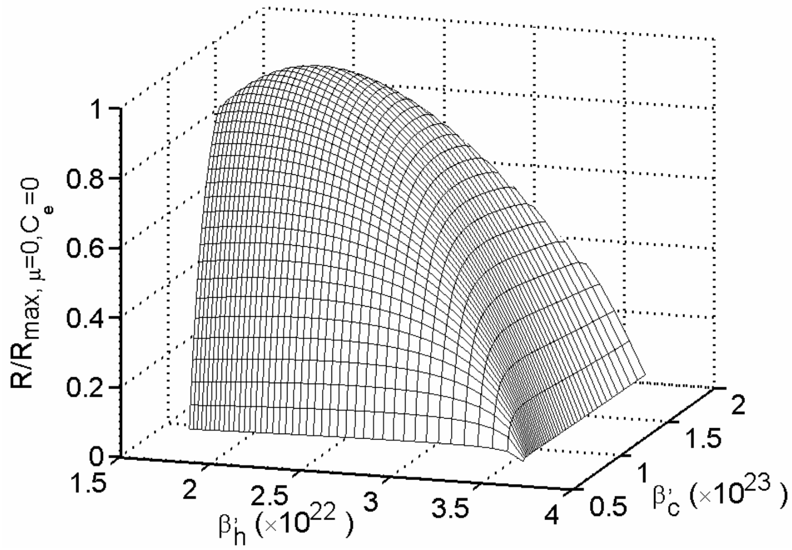


Figure 2. The dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ versus "temperatures" (β'_h, β'_c)

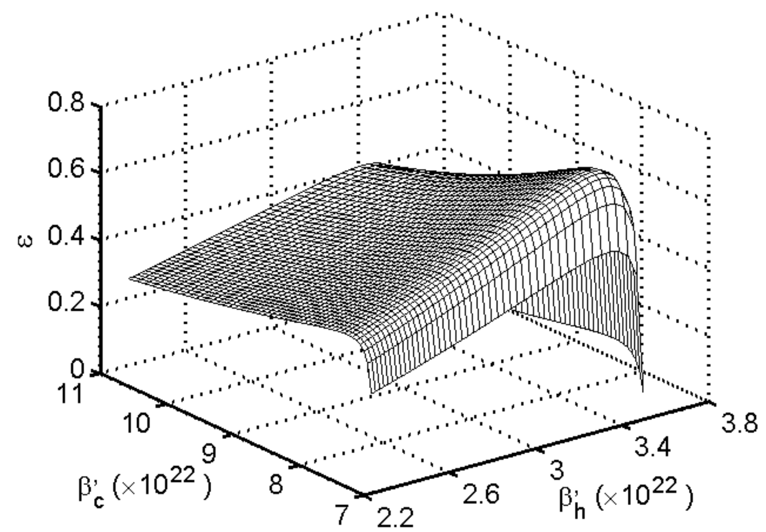


Figure 3. The COP ε versus "temperatures" (β'_h, β'_c)

6. Optimal fundamental characteristic at high temperature limit

When the temperatures of the heat reservoirs and working medium are high enough, i.e. $\hbar\beta\omega \ll 1$, the results of the quantum refrigerator obtained above can be reduced. At the first order approximation $e^z = 1 + z$, equations (29), (30) and (34) can be reduced to

$$Q'_{\rm h} = \frac{\ln(x + \hbar xy\mu^2\omega_3/\tau_{\rm a}) - \ln(\omega_3/\omega_1)}{xy} \quad (38)$$

$$Q'_{\rm c} = \frac{-\ln(\omega_3/\omega_1) - \ln(1/x + \hbar y\mu^2\omega_1/\tau_{\rm b})}{y} \quad (39)$$

$$\dot{Q}_{\rm e} \approx C_{\rm e}[2{\rm c}\hbar\omega_{\rm c}(1 + \lambda\hbar\beta_{\rm h}\omega_{\rm c})/\beta_{\rm c}](\beta_{\rm c} - \beta_{\rm h}) = C_{\rm e}\alpha(\beta_{\rm c} - \beta_{\rm h}) \quad (40)$$

where $x = T'_{\rm c}/T'_{\rm h} = \beta'_{\rm h}/\beta'_{\rm c}$, $y = \beta'_{\rm c}$ and $\alpha = 2{\rm c}\hbar\omega_{\rm c}(1 + \lambda\hbar\beta_{\rm h}\omega_{\rm c})/\beta_{\rm c}$.

At the second order approximation $e^z = 1 + z + 1/(2z^2)$, the cycle period (28) can be reduced to

$$\tau = \frac{\begin{array}{c}\hbar x^2y^2\omega_3(\tau_{\rm a} + \tau_{\rm b})(\mu^2 - 2{\rm a}\tau_{\rm a}\tau_{\rm b}) - \hbar x^2y\beta_{\rm c}\omega_3\tau_{\rm b}[\mu^2 - 2{\rm a}\tau_{\rm a}(\tau_{\rm a} + \tau_{\rm b})] \\ -x^2\beta_{\rm c}\tau_{\rm a}\tau_{\rm b} - \hbar xy\beta_{\rm h}\omega_3\tau_{\rm a}[\mu^2 - 2{\rm a}\tau_{\rm b}(\tau_{\rm a} + \tau_{\rm b})] \\ +x\tau_{\rm a}\tau_{\rm b}[\beta_{\rm c}(\omega_3/\omega_1) + \beta_{\rm h} - 2{\rm a}\hbar\beta_{\rm h}\beta_{\rm c}\omega_3(\tau_{\rm a} + \tau_{\rm b})] - \beta_{\rm h}\tau_{\rm a}\tau_{\rm b}(\omega_3/\omega_1)\end{array}}{2{\rm a}\hbar\omega_3\tau_{\rm a}\tau_{\rm b}x(\beta_{\rm h} - xy)(y - \beta_{\rm c})} \quad (41)$$

Using equations (38)-(41), the cooling load and COP can be reduced to

$$R = \frac{2{\rm a}\hbar\omega_3\tau_{\rm a}\tau_{\rm b}x(\beta_{\rm h} - xy)(y - \beta_{\rm c})[\ln(\omega_1/\omega_3) - \ln(1/x + \hbar y\mu^2\omega_1/\tau_{\rm b})]}{\begin{array}{c}\hbar x^2y^3\omega_3(\tau_{\rm a} + \tau_{\rm b})(\mu^2 - 2{\rm a}\tau_{\rm a}\tau_{\rm b}) - \hbar x^2y^2\beta_{\rm c}\omega_3\tau_{\rm b}[\mu^2 - 2{\rm a}\tau_{\rm a}(\tau_{\rm a} + \tau_{\rm b})] \\ -x^2y\beta_{\rm c}\tau_{\rm a}\tau_{\rm b} - \hbar xy^2\beta_{\rm h}\omega_3\tau_{\rm a}[\mu^2 - 2{\rm a}\tau_{\rm b}(\tau_{\rm a} + \tau_{\rm b})] \\ +xy\tau_{\rm a}\tau_{\rm b}[\beta_{\rm c}(\omega_3/\omega_1) + \beta_{\rm h} - 2{\rm a}\hbar\beta_{\rm h}\beta_{\rm c}\omega_3(\tau_{\rm a} + \tau_{\rm b})] - y\beta_{\rm h}\tau_{\rm a}\tau_{\rm b}(\omega_3/\omega_1)\end{array}} - C_{\rm e}\alpha(\beta_{\rm c} - \beta_{\rm h}) \quad (42)$$

$$\varepsilon = \frac{-x\ln(\omega_3/\omega_1) - x\ln(1/x + \hbar y\mu^2\omega_1/\tau_{\rm b}) - xyC_{\rm e}\alpha(\beta_{\rm c} - \beta_{\rm h})\tau}{x\ln(1/x + \hbar y\mu^2\omega_1/\tau_{\rm b}) + \ln(x + \hbar xy\mu^2\omega_3/\tau_{\rm a}) + (x - 1)\ln(\omega_3/\omega_1)} \quad (43)$$

At high temperature limit, one can find that it is also hard to optimize cooling load R and COP ε and can not obtain the fundamental optimal relation between the cooling load R and COP ε analytically from equations (42) and (43). Therefore, one has to use numerical calculation method in the following analysis and optimization. From equations (42) and (43), one can plot three-dimensional diagrams of dimensionless cooling load ($R/R_{\max,\mu=0,C_{\rm e}=0}$, $\beta'_{\rm h}$, $\beta'_{\rm c}$) and COP (ε, $\beta'_{\rm h}$, $\beta'_{\rm c}$) as shown in Figures 4 and 5, where $R_{\max,\mu=0,C_{\rm e}=0}$ is the maximum cooling load for the endoreversible case at high temperature limit. The parameters used in numerical calculations are ${\rm a} = 2$, ${\rm c} = 2$, $\beta_{\rm h} = 1/(300k_{\rm B})$, $\beta_{\rm c} = 1/(260k_{\rm B})$, $\tau_{\rm a} = 0.01$, $\tau_{\rm b} = 0.01$, $\lambda = -0.5$, $\omega_1 = 1.2\times10^{12}$, $\omega_3 = 1\times10^{11}$, $\omega_{\rm c} = 9\times10^{10}$, $\mu = 0.05$, and $C_{\rm e} = 0.03$. From Figure 4, one can see that there also exists a maximum dimensionless cooling load $(R/R_{\max,\mu=0,C_{\rm e}=0})_{\max}$ for the harmonic quantum Carnot refrigerator. Affected by the internal friction and bypass heat leakage, the maximum dimensionless cooling load $(R/R_{\max,\mu=0,C_{\rm e}=0})_{\max} < 1$. From Figures 3 and 5, one can see that the shape of the three-dimensional diagram of COP (ε, $\beta'_{\rm h}$, $\beta'_{\rm c}$) at high temperature limit is similar to that in general case, and there also exists a maximum COP $\varepsilon_{\max}$ with nonzero corresponding dimensionless cooling load for the harmonic quantum Carnot refrigerator. The optimal "temperature" $\beta'_{\rm h}$ (or $\beta'_{\rm c}$) corresponding to the maximum COP $\varepsilon_{\max}$ is close to the "temperature" of heat reservoirs $\beta_{\rm h}$ (or $\beta_{\rm c}$) at high temperature limit.

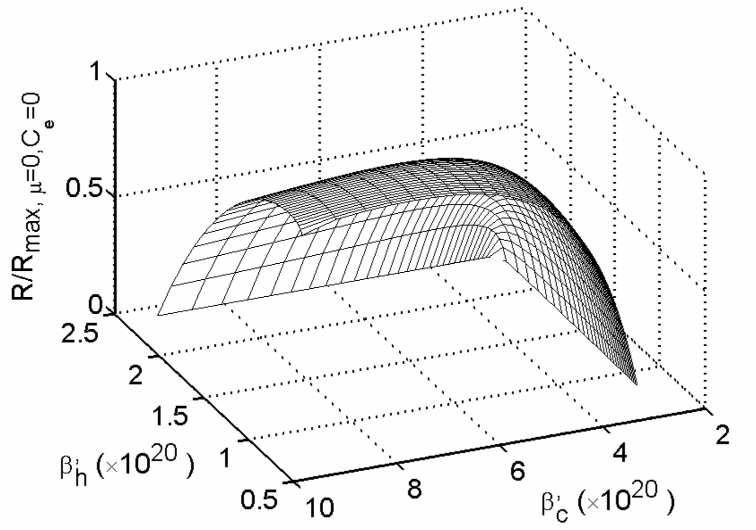


Figure 4. The dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ versus "temperatures" (β_h', β_c') at high temperature limit

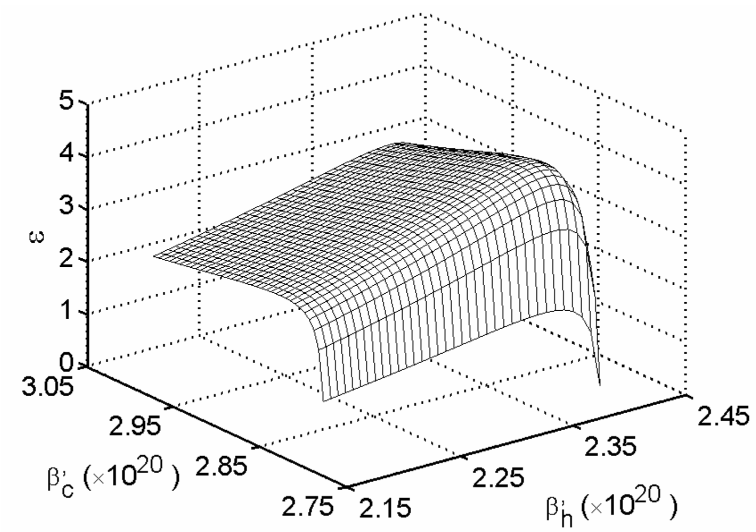


Figure 5. The COP ε versus "temperatures" (β_h', β_c') at high temperature limit

To maximize the cooling load R for fixed COP ε or maximize the COP ε for fixed cooling load R, one can introduce the Lagrangian functions $L_1 = R + \lambda_1\varepsilon$ or $L_2 = \varepsilon + \lambda_2 R$, where λ_1 and λ_2 are two Lagrangian multipliers. Theoretically, solving the Euler-Lagrange equations

$$\partial L_1/\partial x = 0, \quad \partial L_1/\partial y = 0 \tag{44}$$

Or

$$\partial L_2/\partial x = 0, \quad \partial L_2/\partial y = 0 \tag{45}$$

gives the optimal "temperatures" β_h' and β_c'. Combining equations (42) and (43) with the Euler-Lagrange equations above, one can not solve these equations analytically. Solving Euler-Lagrange equations numerically, one can plot the optimal characteristic curves of the dimensionless cooling load versus COP $R/R_{\max,\mu=0,C_e=0} - \varepsilon$, as shown in Figures 6 and 7. Except μ and C_e, the values of the parameters used in the numerical calculations are the same as those used in the numerical calculations of Figure 4. Figures 6 and 7 show that the $R/R_{\max,\mu=0,C_e=0} - \varepsilon$ curves are parabolic-like ones and the dimensionless cooling load has a maximum when there is no bypass heat leakage ($C_e = 0$). When there exists bypass heat leakage ($C_e \neq 0$), the $R/R_{\max,\mu=0,C_e=0} - \varepsilon$ curves are loop-shaped ones, and both the cooling load and COP have maximums. For a fixed bypass heat leakage, both the available cooling load and COP decrease with the increase in the internal friction μ. There are two different COPs for a given cooling load and the quantum refrigerator should work at the point that the COP is higher.

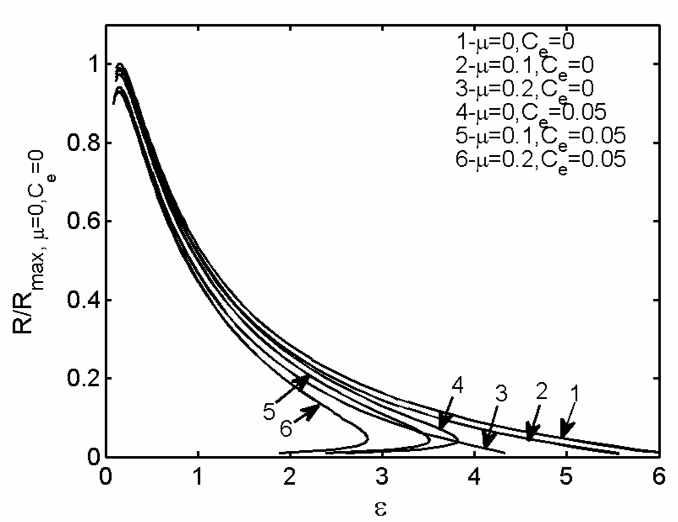


Figure 6. Effects of internal friction μ and bypass heat leakage C_e on dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ versus COP ε

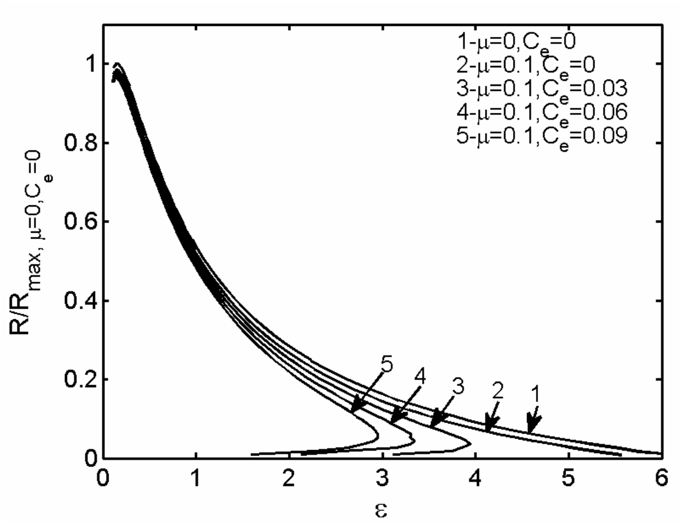


Figure 7. Effects of internal friction μ and bypass heat leakage C_e on dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ versus COP ε

7. Three special cases

The results of this paper include the optimal cooling load and COP characteristics in three special cases, that is, endoreversible case, frictionless case and the case without bypass heat leakage.

(1) The endoreversible case. In this case there is the sole irreversibility of heat resistance in the cycle. Compared to the time spent on the two isothermal processes, the time spent on the two adiabatic processes is negligible (i.e. $\tau_a = \tau_b = 0$), and equations (28), (36) and (37) become

$$\tau = \frac{1}{2a}\int_{\hbar\beta_h'\omega_1}^{\hbar\beta_h'\omega_4} \frac{dm_h}{e^{q\alpha_h m_h}(e^{\alpha_h m_h} - e^{m_h})(1-e^{-m_h})} + \frac{1}{2a}\int_{\hbar\beta_c'\omega_3}^{\hbar\beta_c'\omega_2} \frac{dm_c}{e^{q\alpha_c m_c}(e^{m_c} - e^{\alpha_c m_c})(1-e^{-m_c})} \tag{46}$$

$$R = \frac{1}{\beta'_{\rm c}}[(1+n_3)\ln(1+n_3) - n_3 \ln n_3 - (1+n_1)\ln(1+n_1) + n_1 \ln n_1]\tau^{-1} \tag{47}$$

$$\varepsilon = \frac{x}{1-x} \tag{48}$$

At high temperature limit, equations (46) and (47) can be simplified to

$$\tau = \frac{-x^2\beta_{\rm c} + x\beta_{\rm c}(\omega_3/\omega_1) + x\beta_{\rm h} - \beta_{\rm h}(\omega_3/\omega_1)}{2{\rm a}\hbar\omega_3 x(\beta_{\rm h} - xy)(y-\beta_{\rm c})} \tag{49}$$

$$R = \frac{2{\rm a}\hbar\omega_3 x(\beta_{\rm h} - xy)(y-\beta_{\rm c})\ln(x\omega_1/\omega_3)}{-x^2 y\beta_{\rm c} + xy\beta_{\rm c}(\omega_3/\omega_1) + xy\beta_{\rm h} - y\beta_{\rm h}(\omega_3/\omega_1)} \tag{50}$$

Using equations (48) and (50), one can derive the fundamental optimal relation between cooling load and COP analytically in endoreversible case

$$R = \frac{2{\rm a}\hbar\omega_3\varepsilon(\sqrt{\beta_{\rm h}(1+\varepsilon)} - \sqrt{\beta_{\rm c}\varepsilon})^2 \ln\{\omega\varepsilon_1/[(1+\varepsilon)\omega_3]\}}{[\beta_{\rm h}(1+\varepsilon) - \beta_{\rm c}\varepsilon][\varepsilon - (1+\varepsilon)\omega_3/\omega_1]} \tag{51}$$

For given n_1 and n_3, one can drive the maximum cooling load and corresponding COP of the quantum Carnot refrigerator, and these are the results obtained in Ref. [18].

(2) The frictionless case. In this case there are irreversibilities of heat resistance and bypass heat leakage in the cycle. The time spent on the two adiabatic processes is negligible (i.e. $\tau_{\rm a} = \tau_{\rm b} = 0$), and equations (36) and (37) become

$$R = \frac{1}{\beta'_{\rm c}}[(1+n_3)\ln(1+n_3) - n_3 \ln n_3 - (1+n_1)\ln(1+n_1) + n_1 \ln n_1]\tau^{-1} \\ -2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1-({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}}-1)n_{\rm c}] \tag{52}$$

$$\varepsilon = \frac{\beta'_{\rm h}}{\beta'_{\rm c} - \beta'_{\rm h}} - 2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1-({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}}-1)n_{\rm c}]\tau\{(\frac{1}{\beta'_{\rm h}} - \frac{1}{\beta'_{\rm c}}) \\ \times[(1+n_3)\ln(1+n_3) - n_3 \ln n_3 - (1+n_1)\ln(1+n_1) + n_1 \ln n_1]\}^{-1} \tag{53}$$

The cycle period is independent of bypass heat leakage so that the expression of cycle period in the frictionless case is still equation (46). At high temperature limit, equations (52) and (53) can be simplified to

$$R = \frac{2{\rm a}\hbar\omega_3 x(\beta_{\rm h} - xy)(y-\beta_{\rm c})\ln(x\omega_1/\omega_3)}{-x^2 y\beta_{\rm c} + xy\beta_{\rm c}(\omega_3/\omega_1) + xy\beta_{\rm h} - y\beta_{\rm h}(\omega_3/\omega_1)} - C_{\rm e}\alpha(\beta_{\rm c} - \beta_{\rm h}) \tag{54}$$

$$\varepsilon = \frac{x}{1-x} - \frac{yC_{\rm e}\alpha(\beta_{\rm c} - \beta_{\rm h})[-x^2\beta_{\rm c} + x\beta_{\rm c}(\omega_3/\omega_1) + x\beta_{\rm h} - \beta_{\rm h}(\omega_3/\omega_1)]}{2{\rm a}\hbar\omega_3(\beta_{\rm h} - xy)(y-\beta_{\rm c})(1-x)\ln(x\omega_1/\omega_3)} \tag{55}$$

Using equations (54) and (55), one can not obtain the fundamental optimal relation between the cooling and COP analytically. Using numerical calculations, Figures 6 and 7 show the $R/R_{\max,\mu=0,C_{\rm e}=0} - \varepsilon$ curves (lines 1 and 4 in Figure 6 and line 1 in Figure 7) of the irreversible quantum refrigerator in the frictionless case, and the $R/R_{\max,\mu=0,C_{\rm e}=0} - \varepsilon$ curves are parabolic-like ones, the dimensionless cooling load has a maximum.

(3) The case without bypass heat leakage. In this case, there are irreversibilities of heat resistance and internal friction in the cycle. Equations (36) and (37) become

$$R = \frac{1}{\beta_{\mathrm{c}}'}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2})\tau^{-1} \tag{56}$$

$$\begin{aligned}\varepsilon = {} & [\frac{1}{\beta_{\mathrm{c}}'}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2})][\frac{1}{\beta_{\mathrm{h}}'}(n_1 \ln\frac{n_1}{1+n_1} \\ & -n_4 \ln\frac{n_4}{1+n_4} + \ln\frac{1+n_4}{1+n_1}) - \frac{1}{\beta_{\mathrm{c}}'}(n_3 \ln\frac{1+n_3}{n_3} - n_2 \ln\frac{1+n_2}{n_2} + \ln\frac{1+n_3}{1+n_2})]^{-1}\end{aligned} \tag{57}$$

The cycle period is independent of bypass heat leakage, so that the expression of cycle period of the refrigerator in frictionless case is still equation (28). At high temperature limit, equations (56) and (57) can be simplified to

$$\begin{aligned}R = {} & \frac{2\mathrm{a}\hbar\omega_3\tau_{\mathrm{a}}\tau_{\mathrm{b}}x(\beta_{\mathrm{h}} - xy)(y-\beta_{\mathrm{c}})[\ln(\omega_1/\omega_3) - \ln(1/x + \hbar y\mu^2\omega_1/\tau_{\mathrm{b}})]}{\hbar x^2y^3\omega_3(\tau_{\mathrm{a}}+\tau_{\mathrm{b}})\times(\mu^2 - 2\mathrm{a}\tau_{\mathrm{a}}\tau_{\mathrm{b}}) - \hbar x^2y^2\beta_{\mathrm{c}}\omega_3\tau_{\mathrm{b}}[\mu^2 - 2\mathrm{a}\tau_{\mathrm{a}}(\tau_{\mathrm{a}}+\tau_{\mathrm{b}})]} \\ & -x^2y\beta_{\mathrm{c}}\tau_{\mathrm{a}}\tau_{\mathrm{b}} - \hbar xy^2\beta_{\mathrm{h}}\omega_3\tau_{\mathrm{a}}[\mu^2 - 2\mathrm{a}\tau_{\mathrm{b}}(\tau_{\mathrm{a}}+\tau_{\mathrm{b}})] + xy\tau_{\mathrm{a}}\tau_{\mathrm{b}}[\beta_{\mathrm{c}}(\omega_3/\omega_1) \\ & +\beta_{\mathrm{h}} - 2\mathrm{a}\hbar\beta_{\mathrm{h}}\beta_{\mathrm{c}}\omega_3(\tau_{\mathrm{a}}+\tau_{\mathrm{b}})] - y\beta_{\mathrm{h}}\tau_{\mathrm{a}}\tau_{\mathrm{b}}(\omega_3/\omega_1)\end{aligned} \tag{58}$$

$$\varepsilon = \frac{-x\ln(\omega_3/\omega_1) - x\ln(1/x + \hbar y\mu^2\omega_1/\tau_{\mathrm{b}})}{x\ln(1/x + \hbar y\mu^2\omega_1/\tau_{\mathrm{b}}) + \ln(x + \hbar xy\mu^2\omega_3/\tau_{\mathrm{a}}) + (x-1)\ln(\omega_3/\omega_1)} \tag{59}$$

Using equations (41), (58) and (59), one can drive the maximum cooling load and corresponding COP of the irreversible quantum Carnot refrigerator in the case without bypass heat leakage analytically for given n_1 and n_3.

8. Conclusion

In this paper, an irreversible quantum Carnot refrigerator model with working medium consisting of many non-interacting harmonic oscillators is established. The quantum refrigeration cycle is composed of two isothermal processes and two irreversible adiabatic processes. The irreversibilities of heat resistance, internal friction and bypass heat leakage are considered in the quantum refrigerator model. By using the quantum master equation, semi-group approach and FTT theory, this paper derives the equations of cycle period, cooling load and COP, and provides detailed numerical examples. The numerical examples show that the cooling load has a maximum, and the COP has a maximum with nonzero corresponding dimensionless cooling load when there exists a bypass heat leakage. The optimal performance of the quantum Carnot refrigerator at high temperature limit is derived and analyzed in detail with numerical examples, the optimal characteristic $R/R_{\max,\mu=0,C_{\mathrm{e}}=0} - \varepsilon$ curves are plotted, and effects of internal friction and bypass heat leakage one the optimal performance are discussed. Three special cases, i.e., endoreversible case, frictionless case and the case without bypass heat leakage, are discussed. The numerical examples show that both the cooling load and COP have maximums. The $R/R_{\max,\mu=0,C_{\mathrm{e}}=0} - \varepsilon$ curves are parabolic-like ones and the dimensionless cooling load has a maximum when there is no bypass heat leakage. When there exists bypass heat leakage, the $R/R_{\max,\mu=0,C_{\mathrm{e}}=0} - \varepsilon$ curves are loop-shaped ones. The internal friction decreases the cooling load and COP, but has no effect on the shape of the $R/R_{\max,\mu=0,C_{\mathrm{e}}=0} - \varepsilon$ curves. The obtained results include the fundamental optimal cooling load and COP characteristics in endoreversible case, frictionless case and the case without heat leakage. They are general and can enrich the FTT theory for quantum thermodynamic cycles.

Acknowledgments

This paper is supported by The National Natural Science Foundation of P. R. China (Projects No. 50846040 and 10905093).

References

[1] Andresen B. Finite-Time Thermodynamics. Physics Laboratory II, University of Copenhagen, 1983.
[2] Bejan A. Entropy generation minimization: The new thermodynamics of finite-size devices and finite-time processes. J. Appl. Phys., 1996, 79(3): 1191-1218.

[3] Berry R S, Kazakov V A, Sieniutycz S, Szwast Z, Tsirlin A M. Thermodynamic Optimization of Finite Time Processes. Chichester: Wiley, 1999.
[4] Chen L, Wu C, Sun F. Finite time thermodynamic optimization or entropy generation minimization of energy systems. J. Non-Equilib. Thermodyn., 1999, 24(4): 327-359.
[5] Chen L, Sun F. Advances in Finite Time Thermodynamics: Analysis and Optimization. New York: Nova Science Publishers, 2004.
[6] Chen L. Finite-Time Thermodynamic Analysis of Irreversible Processes and Cycles. Higher Education Press, Beijing, 2005
[7] Feidt M. Optimal use of energy systems and processes. Int. J. Exergy, 2008, 5(5/6): 500-531.
[8] Andresen B. Current trends in finite-time thermodynamics. Angewandte Chemie International Edition, 2011, 50(12) : 2690-2704.
[9] Li J, Chen L, Ge Y, Sun F. Progress in the study on finite time thermodynamic optimization for direct and reverse two-heat-reservoir thermodynamic cycles. Acta Physica Sinica, 2013, 62(13): 130501.
[10] Sieniutycz S, Jezowski J. Energy Optimization in Process Systems and Fuel Cells. 2013, Oxford, UK: Elsevier.
[11] Kosloff R. Quantum thermodynamics: A dynamical viewpoit. Entropy, 2013, 15(6): 2100-2128.
[12] von Spakovsky M R, Gemmer J. Some trends in quantum thermodynamics. Entropy, 2014, 16(6): 3434-3470.
[13] Geva E, Kosloff R. A quantum-mechanical heat engine operating in finite time. A model consisting of spin-1/2 systems as working fluid. J. Chem. Phys., 1992, 96(4): 3054-3067.
[14] Geva E, Kosloff R. On the classical limit of quantum thermodynamics in finite time. J. Chem. Phys., 1992, 97(5): 4398-4412.
[15] Wu F, Sun F, Chen L. Optimum performance of a piston quantum Stirling engine. Pow. Sys. Engne., 1996, 12(1): 41-43 (in Chinese).
[16] Feldmann T, Geva E, Kosloff R. Heat engines in finite time governed by master equations. Am. J. Phys., 1996, 64(4): 485-492.
[17] Wu F, Sun F, Chen L. Optimum performance of a spin-1/2 quantum Carnot refrigerator. Vacuum & Cryogenics, 1996, 2(1): 17-21 (in Chinese).
[18] Wu F, Sun F, Chen L, Hua P. The finite time performance limit and optimization criteria of an quantum Carnot refrigerator at classical limit. Low Temperature and Specialty Gases, 1997(3): 27-33 (in Chinese).
[19] Wu F, Chen L, Sun F, Wu C, Zhu Y. Performance and optimization criteria of forward and reverse quantum Stirling cycles. Energy Convers. Mgmt., 1998, 39(8): 733-739.
[20] Lin B, Chen J. Optimal analysis on the performance of an irreversible harmonic quantum Brayton refrigerator cycle. Phys. Rev. E, 2003, 68(5): 056117.
[21] He J, Xin Y, He X. Performance optimization of quantum Brayton refrigeration cycle working with spin systems. Appl. Energy, 2007, 84(2): 176-186.
[22] Jin X, Wu F, Sun F, Chen L. Exergoeconomic optimal performance of a Carnot quantum engine at classical limit. Pow. Sys. Engng., 1996, 12(6): 42-45 (in Chinese).
[23] Feldmann T, Kosloff R. Performance of discrete heat engines and heat pumps in finite time. Phys. Rev. E, 2000, 61(5): 4774-4790.
[24] Kosloff R, Feldmann T. Discrete four-stroke quantum heat engine exploring the origin of friction. Phys. Rev. E, 2002, 65(2): 055102.
[25] Feldmann T, Kosloff R. Quantum four-stroke heat engine: Thermodynamic observables in a model with intrinsic friction. Phys. Rev. E, 2003, 68(1): 016101.
[26] Sanchez-Salas N, Hernandez A C. Harmonic quantum heat devices: Optimum-performance regimes. Phys. Rev. E, 2004, 70(4): 046134.
[27] Feldmann T, Kosloff R. Characteristics of the limit cycle of a reciprocating quantum heat engine. Phys. Rev. E, 2004, 70(4): 046110.
[28] Wang J, He J, Xin Y. Performance analysis of a spin quantum heat engine cycle with internal friction. Phys. Scr., 2007, 75(2): 227-234.
[29] Rezek Y. Reflections on Friction in quantum mechanics. Entropy, 2010, 12(7): 1885-1901.
[30] Rezek Y, Kosloff R. Irreversible performance of a quantum harmonic heat engine. New J. Phys., 2006, 6(5): 83.

[31] He J, He X, Tang W. The performance characteristics of an irreversible quantum Otto harmonic cycles. Sci. China Ser. G: Phys. Mech. Astron., 2009, 52(9): 1317-1323.

[32] He J, Chen J, Hua B. Quantum refrigeration cycles using spin-1/2 systems as the working substance. Phys. Rev. E, 2002, 65(3): 036145.

[33] Lin B, Chen J, Hua B. The optimal performance of a quantum refrigerator cycle working with harmonic oscillators. J. Phys. D: Appl. Phys., 2003, 36(4): 406-413.

[34] Wu F, Chen L, Sun F, Wu C, Guo F. Optimal performance of an irreversible quantum Brayton refrigerator with ideal Bose gases. Phys. Scr., 2006, 73(5): 452-457.

[35] Wu F, Chen L, Wu S, Sun F. Performance of an irreversible quantum Ericsson cooler at low temperature limit. J. Phys. D: Appl. Phys., 2006, 39(21): 4731-4737.

[36] Liu X, Chen L, Wu F, Sun F. Ecological optimization of an irreversible harmonic oscillators Carnot heat engine. Sci. China Ser. G: Phys. Mech. Astron., 2009, 52(12): 1976-1988.

[37] Liu X, Chen L, Wu F, Sun F. Ecological optimization of an irreversible quantum Carnot heat engine with spin-1/2 systems. Phys. Scr., 2010, 81(2): 025003.

[38] Alicki R, Leudi K. Quantum Dynamic Semi-groups and Applications. Berlin: Springer, 1987.

[39] Zeng J. Quantum Mechanics(3 ed). Beijing: Science Press, 2000 (in Chinese).

[40] Kubo R, Toda M, Hashitsume N. Statistical Physics, Non-equilibrium Statistical Mechanics. Berlin: Springer, 1983.

[41] Ronnie K, Ratner M A, Davis W B. Dynamics and relaxation in interacting systems: Semigroup methods. J. Chem. Phys., 1997, 106(17): 7036-7043.

[42] Davis W B, Wasielewski M R, Ronnie K, Ratner M A. Semigroup representations, site couplings, and relaxation in quantum systems. J. Chem. Phys., 1998, 102(47): 9360-9366.

[43] Wang J, He J, Mao Z. Performance of a quantum heat engine cycle working with harmonic oscillator systems. Sci. China Ser. G: Phys. Mech. Astron., 2007, 50(2): 163-176.

[44] Chen L, Sun F, Wu C, Kiang R L. A generalized model of a real refrigerator and its performance. Appl. Thermal Engng., 1997, 17(4): 401-412.

[45] Chen L, Sun F, Wu C. Effect of heat transfer law on the performance of a generalized irreversible Carnot refrigerator. J. Non-Equilib. Thermodyn., 2001, 26(3): 291-304.

[46] Chen L, Sun F, Wu C. Optimum allocation of heat exchanger area for refrigeration and air conditioning plants. Appl. Energy, 2004, 77(3): 339-354.

[47] Chen L, Zhu X, Sun F, Wu C. Ecological optimization for generalized irreversible Carnot refrigerators. J. Phys. D: Appl. Phys., 2005, 38(1): 113-118.

[48] Zhu X, Chen L, Sun F, Wu C. Exergy based ecological optimization for a generalized irreversible Carnot refrigerator. J. Energy Inst., 2006, 79(1): 42-46.

[49] Li J, Chen L, Sun F. Cooling load and coefficient of performance optimizations for a generalized irreversible Carnot refrigerator with heat transfer law $q^{\alpha}(\Delta T^{n})^{m}$. Proc. IMechE, Part E: J. Process Mech. Eng., 2008, 222(E1): 55-62.

Nomenclature

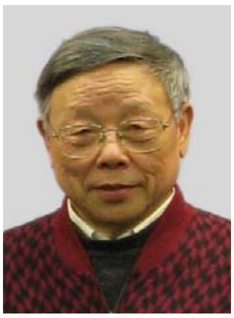
a	parameter of heat reservoir (s^{-1})	t	time (s)
$\hat{a}^{+}$, $\hat{a}$	the Bosonic creation and annihilation operators	W	work (J)
B	heat reservoir	x	"temperature" ratio $x = T_c'/T_h'$
C_e	dimensionless factor which describes the magnitude of the bypass heat leakage	y	"temperature" $\beta = 1/(k_B T')$ (J^{-1})
c	parameter of heat reservoir (s^{-1})	***Greek symbols***	
E	internal energy of the harmonic oscillator system (J)	α	intermediate variable
$\hat{H}$	Hamiltonian	β	"temperature" $\beta = 1/(k_B T)$ (J^{-1})
$\hbar$	reduced Planck's constant ($J \cdot s$)	β'	"temperature" of working medium $\beta' = 1/(k_B T')$ (J^{-1})
k_B	Boltzmann constant (J/K)	γ_+, γ_-	phenomenological positive coefficients
L_1, L_2	Lagrangian functions	ε	coefficient of performance
m	intermediate variable	λ	parameter of the heat reservoir
$\hat{N}$	number operator	λ_1, λ_2	Lagrangian multipliers
n	population of the harmonic oscillators	μ	internal friction coefficient

n_0	initial value of n	$\hat{\Gamma}$	interaction strength operator
n_c	population of the thermal phonons of the cold reservoir	τ	time (s) / cycle period (s)
n_e	asymptotic value of n	ω	thermal phonon frequency (s^{-1}) / harmonic oscillator frequency (s^{-1})
Q	amount of heat exchange (J)	***Subscripts***	
$\hat{Q}_\alpha$, $\hat{Q}_\alpha^+$	operators in the Hilbert space of the system and Hermitian conjugate	B	heat reservoir
$\dot{Q}$	rate of heat flow (W)	c	cold side
Q'	amount of heat exchange between heat reservoir and working medium (J)	h	hot side
q	parameter of heat reservoir	S	working medium system
R	cooling load (W)	SB	interaction between heat reservoir and working medium system
T	absolute temperature (K)	1, 2, 3, 4	cycle states
T'	absolute temperature of the working medium (K)		

Xiaowei Liu received his BS Degree in 2007 in science from Peking University, P R China, and received his MS Degree (2009) and PhD Degree (2013) in power engineering and engineering thermophysics from the Naval University of Engineering, P R China. His work covers topics in quantum thermodynamic cycle and technology support for propulsion plants. Dr Liu is the author or co-author of 20 peer-refereed articles (9 in English journals).

Lingen Chen received all his degrees (BS, 1983; MS, 1986, PhD, 1998) in power engineering and engineering thermophysics from the Naval University of Engineering, P R China. His work covers a diversity of topics in engineering thermodynamics, constructal theory, turbomachinery, reliability engineering, and technology support for propulsion plants. He had been the Director of the Department of Nuclear Energy Science and Engineering, the Superintendent of the Postgraduate School, and the President of the College of Naval Architecture and Power. Now, he is the Direct, Institute of Thermal Science and Power Engineering, the Director, Military Key Laboratory for Naval Ship Power Engineering, and the Dean of the College of Power Engineering, Naval University of Engineering, P R China. Professor Chen is the author or co-author of over 1450 peer-refereed articles (over 640 in English journals) and nine books (two in English).
E-mail address: lgchenna@yahoo.com; lingenchen@hotmail.com, Fax: 0086-27-83638709 Tel: 0086-27-83615046

Feng Wu received his BS Degrees in 1982 in Physics from the Wuhan University of Water Resources and Electricity Engineering, P R China, and received his PhD Degrees in 1998 in power engineering and engineering thermophysics from the Naval University of Engineering, P R China. His work covers a diversity of topics in thermoacoustic engines engineering, quantum thermodynamic cycle, refrigeration and cryogenic engineering. He is a Professor in the School of Science, Wuhan Institute of Technology, P R China. Now, he is the Assistant Principal of Wuhan Institute of Technology, P R China. Professor Wu is the author or co-author of over 160 peer-refereed articles and five books.



Fengrui Sun received his BS Degrees in 1958 in Power Engineering from the Harbing University of Technology, P R China. His work covers a diversity of topics in engineering thermodynamics, constructal theory, reliability engineering, and marine nuclear reactor engineering. He is a Professor in the College of Power Engineering, Naval University of Engineering, P R China. Professor Sun is the author or co-author of over 850 peer-refereed papers (over 440 in English) and two books (one in English).