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Closed-form solution for evaluating the principal instability regions for conservative pipes conveying pulsating flowing fluid

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Abstract
The dynamical behaviors of pipes containing pulsating fluids are nonlinear. In such nonlinear systems principal regions of dynamical instability can result from the effect of the periodic excitation of the fluctuated fluid flow.
Evaluating the boundary frequencies for these types of instabilities had been carried out by many analytical and numerical methods.
In this paper, an approximate “closed-form” solution for evaluating the principal boundary regions of instability for conservative pipes conveying pulsated fluid has been attempted.
Bolotin's method was employed to split the boundaries from the stable regions. The resulting coupled-ordinary differential equations is decoupled by neglecting the effect of the mass ratio term and solved analytically. After imposing the specified boundary conditions, then the simplified formulas or transcendental equations were obtained for the pipes under consideration. These equations can give the principal boundary regions in term of the system parameters by simple calculations.
The present solution was carefully checked with the "exact" solution where the mass ratio was included. The results showed good agreement for pinned-pinned clamped-pinned and clamped-clamped pipes conveying pulsating fluid.

Keywords: Stability boundary; Pulsated fluid; Nonlinear; Vibration; Coriolis; Bolotin; Solution.

1. Introduction

Vibration and stability for pipes conveying steady (un fluctuated) fluid are highly affected by the fluid parameters such as a velocity, pressure, density and viscosity, however, the effect of fluid velocity is the prevail [1]. In the case of conservative pipes conveying a steady fluid such as pinned-pinned, clamped-pinned pipes and clamped-clamped pipes, the natural frequencies are decreased with the increasing of the fluid velocity, until it may reach a critical values making one of the Eigen- values to become null where the pipe loses its stability by buckling [2].
Another type of instability may occur when the fluid velocity fluctuates with the time. In this case the dynamical behaviors become nonlinear and the pipe can lose its stability in "resonance appearance" at

some regions depending on the frequency and amplitude of the excitation parameter. This instability type is called the "dynamical instability" [3].

Boloton [4] introduced periodic series solutions to separate the regions of principal and secondary dynamical instabilities for any elastic systems. The solution with periods equal to twice the natural period of the elastic system can give the principal reigns, and that with period identical to the natural period gives the secondary reigns. The principal instability regions are more important than the secondary since they occur at wider frequency bands.

The dynamics of pipe conveying pulsating fluid was subjected to a considerable attention from many researchers. Early at 1979, Singh and Mallik [5] have studied the problem of pinned-pinned pipe conveying pulsated fluid. He employed Bolotin concept to separate the dynamic instability regions and solved the resulted coupled differential equations by using an exponential series solution.

Later, Wang and Bloom [6] studied the parametric instability for concentrated pipes conveying pulsating fluid by using Bolotin method. They employed power series solution to find the regions of instabilities.

Recently, Wang [7] was further investigated pinned-pinned pipes conveying pulsating fluid by considering the effect of motion constraints. The equation of motion was discretized by using two mode Galerkin method and then solved numerically by using the fourth order Runge–Kutta scheme. He considered the effect of increasing of the mean velocity of the fluctuated flow on the global dynamics.

In the work by Panda and Kar [8] a study of the nonlinear planar vibration of a pipe conveying pulsated fluid was presented. They showed that the system can be subjected to principal parametric resonance in the presence of internal resonance. They analyzed pinned-pinned pipe with geometric cubic nonlinearity. They linearized their model by using the method of multiple scales (MMS).

Liu and Huang [9] introduced a variational method for the formulation of the dynamic instability assessment of axially translating media such as moving strings and pipe conveying fluid. They analyzed the parametric instability problem by reducing it to a stationary value problem in the form of a classical Rayleigh quotient.

Jensen [10] analyzed stability and nonlinear dynamics of articulated cantilever pipes (two segments rigid pipes with flexible joints) conveying fluid with a high-frequency pulsating component. He introduced an approximated nonlinear solution for small-amplitude flutter oscillations by using a fifth-order multiple scales perturbation method.

Kochupillai [11] introduced Finite element method to examine the parametric instability for thin pipe containing pulsating fluid flow by using multivariable Floquet–Lyapunov theory for shells.

Zsolt Szabz [12] introduced the method of multiple scales to analyze the global nonlinear dynamics from the local dynamics of cantilever pipe. The influence of periodic components of pulsating fluid on the dynamics was considered.

As one can see from the above survey that, there existed several analytical and numerical methods of solutions to investigate the dynamical instability arises from the effect of the pulsated fluid. Unfortunately these methods are normally lead to numerical cumbersome calculations.

In the present work an approximated analytical approach will be adopted to evaluate the principal boundaries of dynamical instability. These solutions lead to either a explicit formula in case of pinned-pinned pipes or to simple transcendental equations for clamped-pinned and clamped-clamped pipes.

2. Theoretical consideration

The derivation of the equation of small motion for a pipe conveying unsteady fluid based on the beam theory can be found in many references. For example [5, 6, 13], which takes the following form:

$$EI\frac{\partial^4 y}{\partial x^4}+(m_f V^2+PA_p)\frac{\partial^2 y}{\partial x^2}+2m_f V\frac{\partial^2 y}{\partial x\partial t}+m_f\frac{\partial V}{\partial t}\frac{\partial y}{\partial x}+(m_f+m_p)\frac{\partial^2 y}{\partial t^2}=0 \qquad (1)$$

It is convenient to rewrite Eq.(1) in the following non-dimensional form:

$$\eta^{IV}+(U^2+\gamma)\eta''+2\beta U\dot{\eta}'+\dot{U}\eta'+\ddot{\eta}=0 \qquad (2)$$

where

$\eta = y/L$, $\zeta = x/L$, $U = VL(m_f/EI)^{1/2}$, $\gamma = P A_p L^2/EI$, $\beta = [m_f/(m_f + m_p)]^{1/2}$.

τ=(t/L2)[EI/(mf+mp)]1/2 (3)

The notations (') and (.) are means ∂Y/∂ζ and ∂Y/∂τ, respectively.
Consider the fluid velocity pulsates at frequency (Ω) and amplitude (δ), around its mean value of (Uo). The mathematical representation for this fluctuated velocity is:

U=Uo(1+δcosΩτ) (4)

Substituting eq. (4) into eq. (2) giving:

η IV+[Uo2(1+δcosΩτ)2+γ] η ″+2βUo (1+δcosΩτ) $\dot{\eta}'$ - δ Uo Ω sin Ω τ η' + $\ddot{\eta}$ = 0 (5)

2.1 The principal regions of instability
Eq.(5) is a non-linear partial differential equation with periodic coefficients (Mathieu-Hill Type). The complete solution for such an equation is not available in the literature, however many spatial solutions were available to find some important behaviors. For example Bolotin's solution which is useful for evaluating the boundaries of dynamical instability regions.
To separate the principal regions of instability from the stable ones by using Bolotin's solution the following series solution can be used [4]:

$$\eta\,(\zeta,\tau) = \sum_{k=1,3,5,...}^{\infty} X_k(\zeta)\sin(k\Omega\tau/2) + Y_k(\zeta)\cos(k\Omega\tau/2) \quad (6)$$

Substituting eq. (6) solution into eq. (5) resulting an infinite series according to the infinite values of the index (k). Bolotin has also showed that this series is rapidly converges, and when one term is taken (k =1) a satisfactory results can be obtained. Doing this and equating coefficients of sin (Ωτ/2), and cos (Ωτ/2) the results (see appendix-A) are:

$$X_1^{IV}+(U_o^2+\gamma+\delta^2U_o^2/2-\delta U_o^2)\,X_1''-1/4\Omega^2 X_1-\beta\,U_o\;\Omega\,Y'_1=0 \quad (7)$$

$$Y_1^{IV}+(U_o^2+\gamma+\delta^2U_o^2/2+\delta U_o^2)\,Y_1''-1/4\Omega^2 Y_2+\beta\,U_o\,\Omega\,X'_1=0 \quad (8)$$

For the purpose of comparison two approaches of solutions will be attempted to solve eq.(7) and (8) ;the first is the "exact" solution in which the effect of β is considered and the second is the presented solution where β is neglected.

2.1.1 The "exact" solution
Eqs.(7) and (8) are two ordinary coupled-differential equations and there solutions give the upper and lower boundaries of principal instability regions.
Let the solution be in the following form [5]:

$$X_1=\sum_{j=1}^{8} C_{1j}e^{\lambda j\zeta} \quad \text{and} \quad Y_1=\sum_{j=1}^{8} C_{2j}e^{\lambda j\zeta} \quad (9)$$

where, λ_1……, λ_8 are the roots of the eighth order polynomial resulting from expanding the following determinate equation:

$$\begin{vmatrix} \lambda^4+(U_o^2+\gamma+1/2\delta^2U_o^2-\delta U_o^2)\lambda^2-1/4\Omega^2 & -\beta U_o\Omega\lambda \\ \beta U_o\Omega\lambda & \lambda^4+(U_o^2+\gamma+1/2U_o^2\delta^2+\delta U_o^2)\lambda^2-1/4\Omega^2 \end{vmatrix}=0 \quad (10)$$

And C_{1j} 's and C_{2j} 's are arbitrary constants which are related to each other from application of eq. (9) on any of eqs. (7) and (8).
The boundary conditions for the three pipes in consideration are as the follow:

For pinned-pinned pipe:

$$X_1(0,\tau) = 0,\ X_1''(0,\tau) = 0,\ X_1(1,\tau) = 0,\ X_1''(1,\tau) = 0,$$
$$Y_1(0,\tau) = 0, Y_1''(0,\tau) = 0, Y_1(0,\tau) = 0, Y_1''(1,\tau) = 0 \tag{11}$$

For clamped-pinned pipe:

$$X_1(0,\tau) = 0,\ X_1'(0,\tau) = 0,\ X_1(1,\tau) = 0,\ X_1''(1,\tau) = 0,$$
$$Y_1(0,\tau) = 0, Y_1'(0,\tau) = 0, Y_1(1,\tau) = 0, Y_1''(1,\tau) = 0 \tag{12}$$

And for clamped-clamped pipes:

$$X_1(0,\tau) = 0,\ X_1'(0,\tau) = 0,\ X_1(1,\tau) = 0,\ X_1'(1,\tau) = 0,$$
$$Y_1(0,\tau) = 0, Y_1'(0,\tau) = 0, Y_1(1,\tau) = 0, Y_1'(1,\tau) = 0 \tag{13}$$

For any of the considered pipe supporting, application of the boundary conditions on eq. (9) will lead to the following matrix equation:

$$\mathbf{[A]}\ \mathbf{\{C\}} = 0 \tag{14}$$

Where: **[A]** is 8x8 matrix and **{C}** is 8x1 vector of the constants.
The elements of **[A]** are depending on the boundary conditions, so that there are three forms for **[A]** due to the three pipe supporting configurations, for example **[A]** for pinned-pinned pipe is given in Appendix-B.
For nontrivial solution the following equation must be satisfied,

$$|\mathbf{A}| = 0 \tag{15}$$

The values of (Ω), and (δ) satisfying eq. (15) give the upper and lower boundaries of principal instability regions for the pipe in consideration. In solving eq. (15) a computer codes were written in MATLAB software.

2.1.2 The present technique

It had been demonstrated by many researchers that the effect of the mass ratio β on the natural frequencies for the conservative pipes conveying fluid is insignificant for instant see Long [1], Housner [14], Belevins [15] or Yi-min [16].
From the above conclusion and with keeping in mined that the principal boundaries are a band of resonance frequencies which are strictly depend on the natural frequencies, one can conclude that neglecting of β in the analysis of the principal instability can lead to a reasonable results. This is the core idea of the present approach.
Now with neglecting β eqs. (7) and (8) become,

$$X_1^{IV} + (U_o^2 + \gamma + \delta^2 U_o^2/2 - \delta\, Uo^2)\, X_1'' - ¼\Omega^2 X_1 = 0 \tag{16}$$

$$Y_1^{IV} + (U_o^2 + \gamma + \delta^2 U_o^2/2 + \delta\, U_o^2)\, Y_1'' - ¼\Omega^2 Y_1 = 0 \tag{17}$$

Eqs. (16) and (17) now decoupled and can be solved independently by using the following General solutions,

$$X_1 = A \sin \sigma_1 \zeta + B \cos \sigma_1 \zeta + C \sinh \sigma_2 \zeta + D \cosh \sigma_2 \zeta \tag{18}$$

$$Y_2 = E \sin \sigma_3 \zeta + F \cos \sigma_3 \zeta + G \sinh \sigma_4 \zeta + H \cosh \sigma_4 \zeta \tag{19}$$

where A,B,C,D,E,F,G and H are arbitrary constants depending on the boundary conditions and,

$\sigma_1 = [1/2(R_1 + \sqrt{R_1^2 + \Omega^2})]^{1/2}$,

$\sigma_2 = [1/2(-R_1 + \sqrt{R_1^2 + \Omega^2})]^{1/2}$

$\sigma_3 = [1/2(R_2 + \sqrt{R_2^2 + \Omega^2})]^{1/2}$,

$$\sigma_4 = [1/2(-R_2 + \sqrt{R_2^2 + \Omega^2})]^{1/2} \quad (20)$$

and,

$$R_1 = Uo^2 + \gamma + \delta^2 Uo^2/2 - \delta\, Uo^2,$$
$$R_2 = Uo^2 + \gamma + \delta^2 Uo^2/2 + \delta\, Uo^2 \quad (21)$$

For pinned-pinned pipe the application of the boundary conditions given in eq. (11) on the solutions given in eqs. (18) and (19) resulting the following simplified closed-form solutions,

$$\Omega = 2n\pi\sqrt{(n\pi)^2 - U_O^2 - \gamma - U_O^2(0.5\delta^2 - \delta)}$$
$$\Omega = 2n\pi\sqrt{(n\pi)^2 - U_O^2 - \gamma - U_O^2(0.5\delta^2 + \delta)} \quad (22)$$

Where (n) is the mode number (n =1,2……∞).
eq. (22) give the upper and lower frequency boundaries of principal instability regions at any modes of pinned-pinned pipes conveying fluid.
For clamped-pinned pipe the boundary conditions given in eq. (12) together with eqs. (18) and (19) result the following transcendental equations.

$$\sigma_1 \tanh \sigma_2 - \sigma_2 \tan \sigma_1 = 0,$$
$$\sigma_3 \tanh \sigma_4 - \sigma_4 \tan \sigma_3 = 0 \quad (23)$$

Where σ_1, σ_2, σ_3 and σ_4 were defined in eqs. (20) and (21).
Noting that in this case the upper and lower frequency boundaries of principal instability regions can be found by solving the first and second equations of (23), respectively.
Finally, for clamped-clamped pipes the boundary conditions given in eq. (13), with eqs. (18) and (19) give the following transcendental equations,

$$1 - \cos \sigma_1 \cosh \sigma_2 + 1/2(\sigma_2/\sigma_1 - \sigma_1/\sigma_2)\sin \sigma_1 \sinh \sigma_2 = 0,$$
$$1 - \cos \sigma_3 \cosh \sigma_4 + 1/2(\sigma_3/\sigma_4 - \sigma_4/\sigma_3)\sin \sigma_3 \sinh \sigma_4 = 0 \quad (24)$$

where σ_1, σ_2,σ_3 and σ_4 were defined in eqs. (20) and (21)
The upper and lower frequency boundaries of principal instability regions can be found by solving the first and second of equations (24), respectively.

3. Results and discussions
To examine the validity of the present approach the principal boundaries obtained by the "exact" and the present solution for the specified pipe parameter are plotted. Such plots for the lowest two modes of pinned-pinned, clamped-clamped and clamped-clamped pipes are shown in Figures 1 to 6. Accomplishing the effect of β at nearly full rang the results at β =0, 0.45 and 0.9 are selected (the extreme values of β are 0 and 1).
For all curves one can see the V shape curves which are represents the Bolotin solutions of the dimensionless frequency equations of the principal boundaries given in the theory for example Figure 1 gives three V shapes plots each for the corresponding values of β. The left portion is from plotting of the first of eq. 22 and the right is from the second. The vales of the excitation parameters δ ares chosen from (0 to 0.5).

It is important to note that, in all cases, when β =0 the two solutions are coincided. For the other values of β there are some drift between the two solutions. The percentage error due to these drifts are calculated and presented in Table 1. From Table 1, it is clear that the present solution is in a very good agreement with the "exact" solution in case of the pinned-pinned and clamped- pinned pipes where the maximum error is 3.71%. However, for clamped-clamped pipes the two solutions has somewhat discrepancy at the first mode causing an error of 13.42%, while this error reduces to 3.18%. This may be due of the nature of the clamped two end boundary conditions which increases the axial strain to the pipe and affect the mode shape which is assumed as a lateral only in the theory. However when at least on end is pinned the permitted rotation reduces this effect.

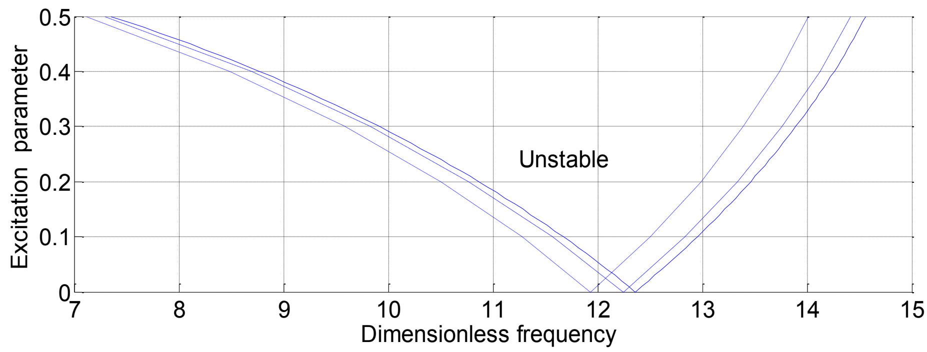

Figure 1. Comparison between the present and the "exact" solution, first mode pinned-pinned pipe. γ=0, Uo=2. (—) present and "exact" for β=0, (– –)"exact" for β=0.45, (– ·) "exact " for β=0.9.

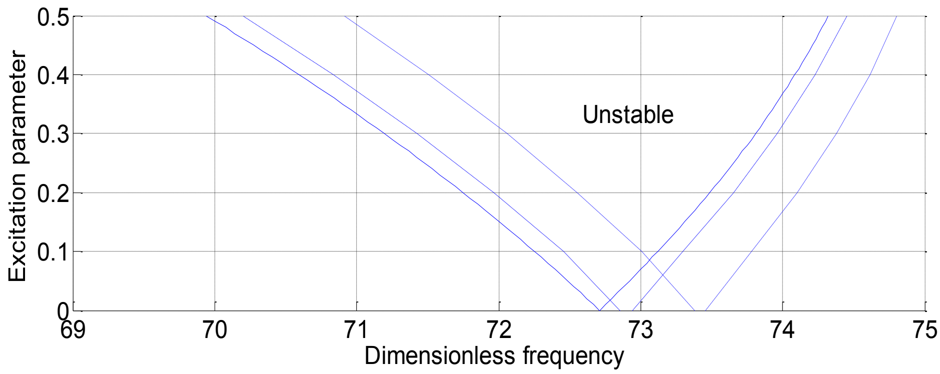

Figure 2. Comparison between the present and the "exact" solution, second mode pinned-pinned pipe γ=0, Uo=2. (—) present and "exact" for β=0, (– –)"exact" for β=0.45, (– ·) "exact" for β=0.9.

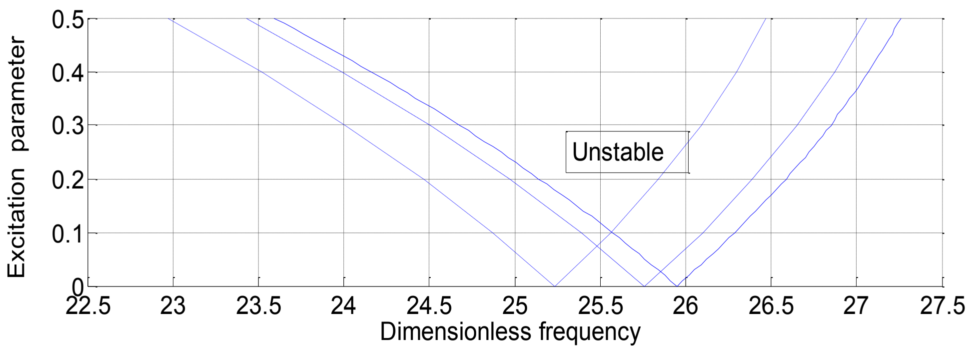

Figure 3. Comparison between the present and the "exact" solution, first mode clamped-pinned pipe. γ=0,Uo=2. (—) present and "exact" for β=0, (– –)"exact" for β=0.45, (– ·) "exact" for β=0.9

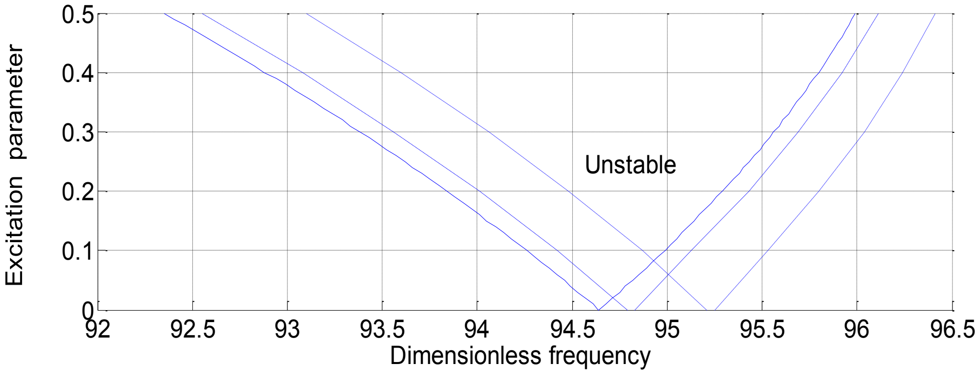

Figure 4. Comparison between the present and the "exact" solution, second mode clamped-pinned pipe γ=0, Uo=2. (—) present and "exact" for β=0, (– –)"exact" for β=0.45, (– ·) "exact" for β=0.9.

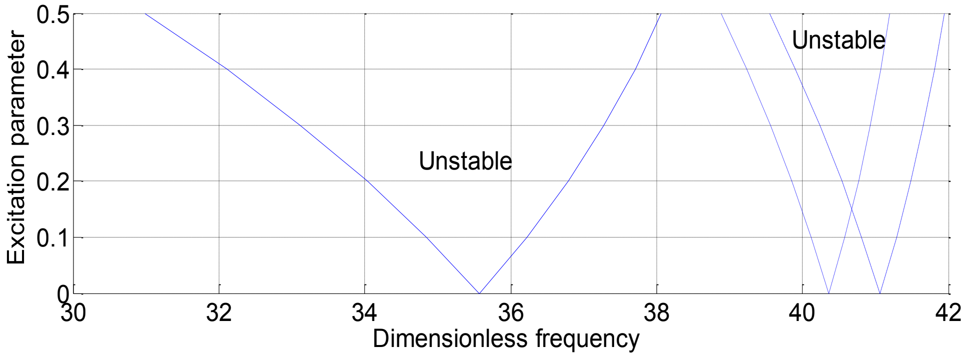

Figure 5. Comparison between the present and the "exact" solution, first mode clamped-clamped pipe γ=0, Uo=2. (—) present and "exact" for β=0, (– –)"exact" for β=0.45, (– ·) "exact" for β=0.9.

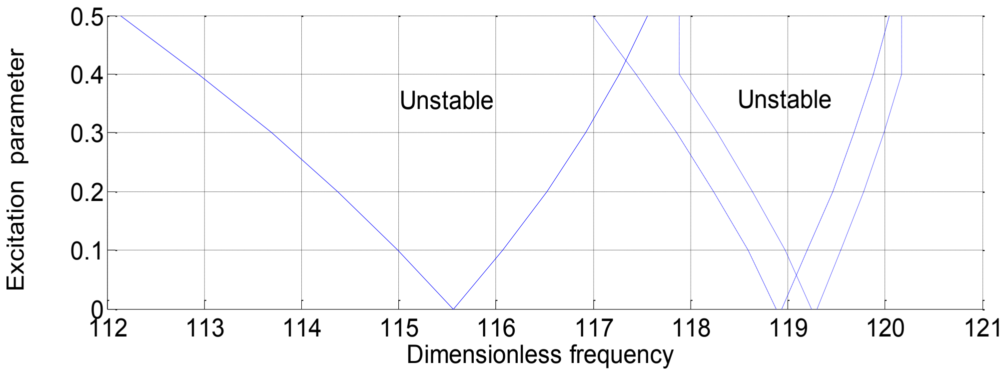

Figure 6. Comparison between the present and the "exact" solution, second mode clamped-pinned pipe γ=0, Uo=2. (—) present and "exact" for β=0, (– –)"exact" for β=0.45, (– ·) "exact" for β=0.9.

Table 1. Percentage error between the "exact "and presented solutions.

Boundary conditions	First mode	Second mode
Pinned-pinned	3.71%	1.41%
Clamped-pinned	2.84%	0.36%
Clamped-clamped	13.42%	3.18%

The presented solution is employed to find the principal instability regions for the lowest three modes for the three pipes in considerations as it is shown in Figures7-9. In general these figures show that these regions are become wider as the excitation parameter δ increases. The tips of the regions are coincided with the double of the natural frequencies for example for pinned-pinned pipes the first three natural frequencies are 6.18, 36.35 and 85.78, [16].

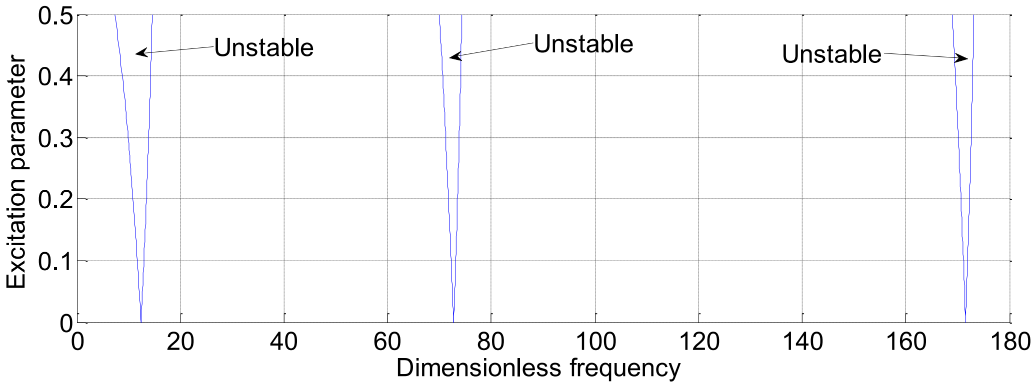

Figure 7. The principal instability regions for the lowest three modes for pinned-pinned pipe. $\gamma=2, Uo=2$.

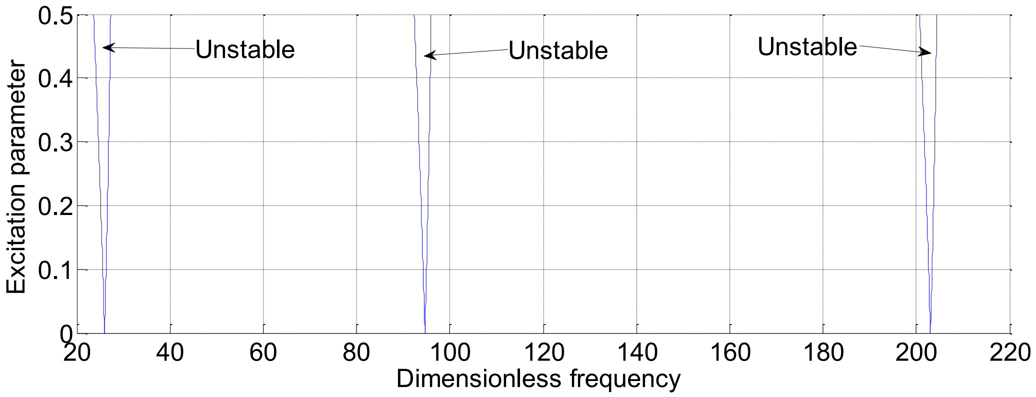

Figure 8. The principal instability regions for the lowest three modes for clamped-pinned pipe, $\gamma=2,, Uo=2$.

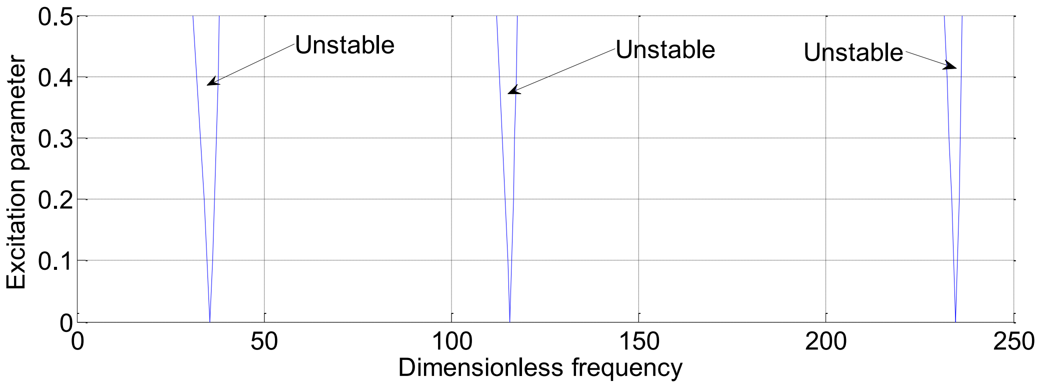

Figure 9. The principal instability regions for the lowest three modes for clamped-clamped pipe, $\gamma=2,, Uo=2$.

The effects of increasing the fluid pressure in the pipes on the principal regions for the three pipes are investigated in Figures 10-12. As it is clear from these figures that are in all pipes, the increasing of the dimensionless pressure γ tends to shift the regions to a lower frequency bands and the regions become wider. These two effects denote that increasing the pressure will force the pipe to operate in more dangerous conditions.

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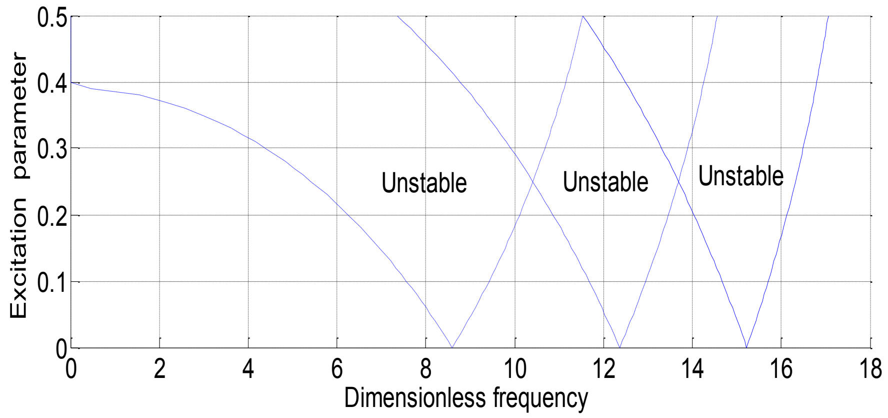

Figure 10. Effect γ on the principal instability regions for the first mode pinned-pinned pipe
(—) γ=0, (– –) γ =2,(– ·) γ =4.

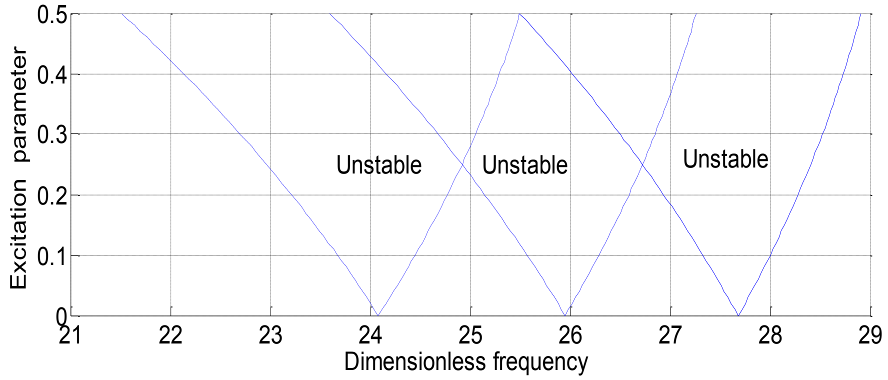

Figure 11. Effect γ on the principal instability regions for the first mode for clamped-pinned pipe
(—) γ=0, (– –) γ =0.2,(– ·) γ =4.

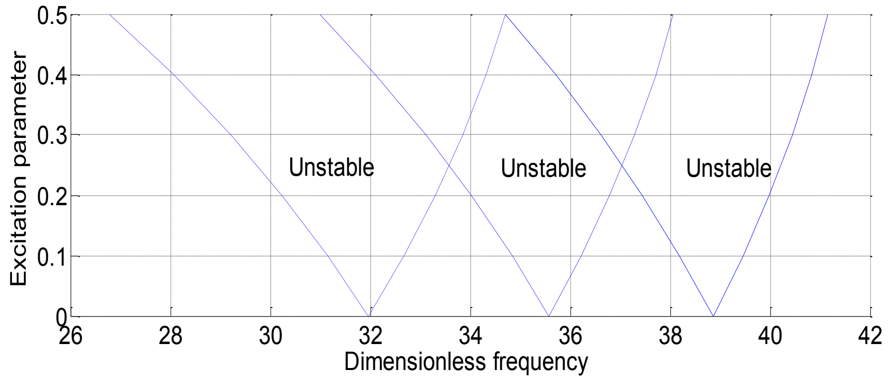

Figure 12. Effect γ on the principal instability regions for the first mode for clamped-clamped pipe
(—) γ=0, (– –) γ =0.2,(– ·) γ =4.

4. Conclusions

1. The present approach provides a very simple and confidence method for evaluating the principal boundary regions of instabilities for conservative pipes conveying pulsated fluid. For pinned-pinned pipes the solutions gives explicit formulas while for clamped-pinned and clamped-clamped pipes it gives a simple transcendental equations. With these solutions the calculation labor and time are exaggeratedly lowered as compare with the other available analytical or numerical methods.

2. The present approach is compared with other solution. The results show a very good agreement for pinned-pinned and clamped-pinned where the maximum error is as small as 3.71% and fair agreement for clamped-clamped where the error is not exceeded the 13.71%.
3. The effect of pressure is investigated by this work. It is found that in case of the pressurized pipes conveying pulsated fluid the parametric instability become more dangerous since it is shifted to a lower frequency bands and become wider as the excitation parameter increases.

Nomenclatures

$A_{f,}A_p$: Fluid and pipe cross sectional area, respectively (m^2).
E: Modulus of elasticity (N/m^2).
L: Pipe length (m).
m_f, m_p: Fluid and pipe mass per unit length, respectively (kg/m).
p: Fluid pressure (N/m^2).
V: Fluid velocity (m/s).
Uo, β, γ: Dimensionless mean velocity, mass ratio, and dimensionless pressure, respectively.
Ω: Dimensionless frequency = $\omega L^2[(m_f + m_p) / E\ I)^{1/2}$.
ω: Circular frequency (rad/sec).
δ: Dimensionless excitation parameter.
τ: Dimensionless time.
Uo: Dimensionless mean speed.

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Appendix A.

Substituting the solution given in eq. (6) into eq. (5) leads to the following series:

$$\sum_{k=1,3,5,\ldots}\Bigg[\left\{\frac{d^4\bar{X}_k}{d\xi^4}+\left(u_0^2+\gamma+\frac{u_0^2\delta^2}{2}\right)-\left(\frac{k}{2}\right)^2\Omega^2\bar{X}_k-\beta u_0\Omega k\frac{d\bar{Y}_k}{d\xi}\right\}\sin(\tfrac{1}{2}k\Omega\tau)$$
$$+\left\{\frac{d^4\bar{Y}_k}{d\xi^4}+\left(u_0^2+\gamma+\frac{u_0^2\delta^2}{2}\right)\frac{d^2\bar{Y}_k}{d\xi^2}-\left(\frac{k}{2}\right)^2\Omega^2\bar{Y}_k+\beta u_0\Omega k\frac{d\bar{X}_k}{d\xi}\right\}\cos(\tfrac{1}{2}k\Omega\tau)$$
$$+\left\{-\left(\frac{k\Omega}{2}\right)\beta u_0\delta\frac{d\bar{Y}_k}{d\xi}+u_0^2\delta\frac{d^2\bar{X}_k}{d\xi^2}-\tfrac{1}{2}\beta u_0\delta\Omega\frac{d\bar{Y}_k}{d\xi}\right\}\sin\left(\frac{k+2}{2}\Omega\tau\right)$$
$$+\left\{\left(\frac{k\Omega}{2}\right)\beta u_0\delta\frac{d\bar{X}_k}{d\xi}+u_0^2\delta\frac{d^2\bar{Y}_k}{d\xi^2}+\tfrac{1}{2}\beta u_0\delta\Omega\frac{d\bar{X}_k}{d\xi}\right\}\cos\left(\frac{k+2}{2}\Omega\tau\right)$$
$$+\left\{-\left(\frac{k\Omega}{2}\right)\beta u_0\delta\frac{d\bar{Y}_k}{d\xi}+u_0^2\delta\frac{d^2\bar{X}_k}{d\xi^2}+\tfrac{1}{2}\beta u_0\delta\Omega\frac{d\bar{Y}_k}{d\xi}\right\}\sin\left(\frac{k-2}{2}\Omega\tau\right)$$
$$+\left\{\left(\frac{k\Omega}{2}\right)\beta u_0\delta\frac{d\bar{X}_k}{d\xi}+u_0^2\delta\frac{d^2\bar{Y}_k}{d\xi^2}-\tfrac{1}{2}\beta u_0\delta\Omega\frac{d\bar{X}_k}{d\xi}\right\}\cos\left(\frac{k-2}{2}\Omega\tau\right)$$
$$+\frac{u_0^2\delta^2}{4}\frac{d^2\bar{X}_k}{d\xi^2}\left\{\sin\left(\frac{k+4}{2}\Omega\tau\right)+\sin\left(\frac{k-4}{2}\Omega\tau\right)\right\}$$
$$+\frac{u_0^2\delta^2}{4}\frac{d^2\bar{Y}_k}{d\xi^2}\left\{\cos\left(\frac{k+4}{2}\Omega\tau\right)+\cos\left(\frac{k-4}{2}\Omega\tau\right)\right\}\Bigg]=0.$$

If this series is truncated at *k=1* and the coefficients of sin and cos are equated the results are eqs. (7) and (8),

B:-for pinned-pinned pipe **[A]** is:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 & \lambda_7^2 & \lambda_8^2\\
e^{\lambda 1} & e^{\lambda 2} & e^{\lambda 3} & e^{\lambda 4} & e^{\lambda 5} & e^{\lambda 6} & e^{\lambda 7} & e^{\lambda 8}\\
\lambda_1^2 e^{\lambda 1} & \lambda_2^2 e^{\lambda 2} & \lambda_3^2 e^{\lambda 3} & \lambda_4^2 e^{\lambda 4} & \lambda_5^2 e^{\lambda 5} & \lambda_6^2 e^{\lambda 6} & \lambda_7^2 e^{\lambda 7} & \lambda_8^2 e^{\lambda 8}\\
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8\\
K_1\lambda_1^2 & K_2\lambda_2^2 & K_3\lambda_3^2 & K_4\lambda_4^2 & K_5\lambda_5^2 & K_6\lambda_6^2 & K_7\lambda_7^2 & K_8\lambda_8^2\\
K_1 e^{\lambda 1} & K_2 e^{\lambda 1} & K_3 e^{\lambda 1} & K_4 e^{\lambda 4} & K_5 e^{\lambda 5} & K_6 e^{\lambda 6} & K_7 e^{\lambda 7} & K_8 e^{\lambda 8}\\
K_1\lambda_1^2 e^{\lambda 1} & K_2\lambda_2^2 e^{\lambda 2} & K_3\lambda_3^2 e^{\lambda 3} & K_4\lambda_4^2 e^{\lambda 4} & K_5\lambda_5^2 e^{\lambda 5} & K_6\lambda_6^2 e^{\lambda 6} & K_7\lambda_7^2 e^{\lambda 7} & K_8\lambda_8^2 e^{\lambda 8}
\end{bmatrix}$$

Where:

$\lambda_1, \ldots, \lambda_8$ are the roots of the following polynomial equation given in eq. (10):

and:

$K_j = \beta\ Uo\ \Omega\ \lambda_j / (\lambda_j^4 + R_1 \lambda_j^2 - 1/4\Omega^2)$, $(j = 1\ \text{to}\ 8)$

$\{Cj\}^T = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$